Chapter I: Basic aspects of cont. time MCs.

1. Markov property: A sequence of rv's is called a (stochastic) process. It is said to have the MP if the future are independent given the present.

The process $(X_n)_{n\in\mathbb{N}}$ is called a discrete-time MC with state space I, if $\forall x_1, \dots, x_n \in I$, we have
$$\mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}, \dots, X_1 = x_1, X_0 = x_0) = \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}).$$

If $\mathbb{P}(X_n = y \mid X_{n-1} = x)$ is indep. of n, the MC is called time-homogeneous. We then write $P = (P_{xy})_{x,y\in I}$ for the transition matrix, i.e. $P_{xy} = \mathbb{P}(X_n = y \mid X_{n-1} = x)$

It is a stochastic matrix because
$$\sum_y P_{xy} = 1 \ \& \ P_{xy} \geq 0.$$

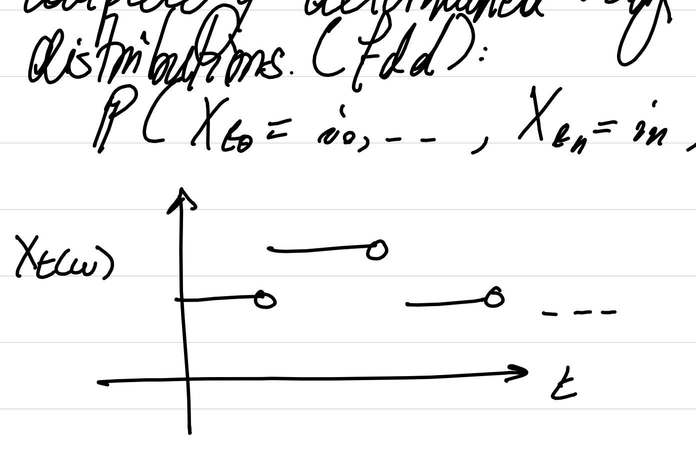
The two things associated with a time-homogeneous MC are —
(I) The initial distribution μ_0, where $\mu_0(x) = \mathbb{P}(X_0 = x)$ $\forall x \in I$.
(II) Transition matrix P.

From now on, (I) I is countable (incl. finite)
(II) $(\Omega, \mathcal{F}, \mathbb{P})$ is a prob. space on which all relevant random variables are defined.

Def.: $X = (X_t : t \geq 0)$ is a right-continuous, continuous-time random process with values in I if —

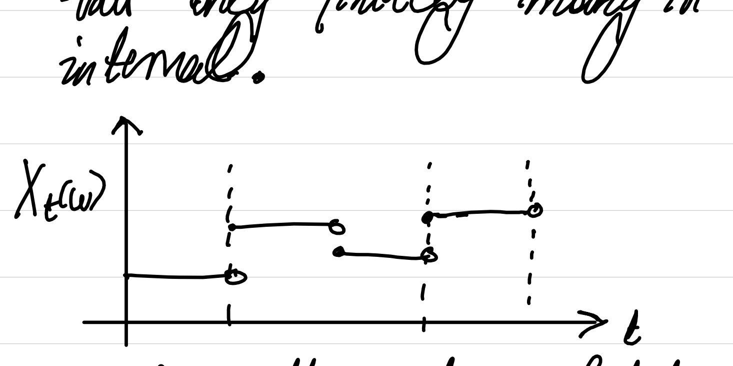
(I) $\forall t \geq 0$ X_t is a random variable with values in I (i.e. $X_t : \Omega \to I$).
(II) $\forall \omega \in \Omega$: $X_\cdot(\omega) : [0,\infty) \to I$ is right-cont. (i.e., has right-continuous sample paths).
So $\forall \omega \in \Omega, t \geq 0, \exists \varepsilon > 0$ s.t. $X_t(\omega) = X_s(\omega)$ $\forall s \in [t, t+\varepsilon]$.

Fact (without proof): A right-cont. process is completely determined by its finite-dim. distributions (fdd):
$$\mathbb{P}(X_{t_0} = i_0, \dots, X_{t_n} = i_n), \ n \in \mathbb{N}, t_k \geq 0, i_k \in I.$$

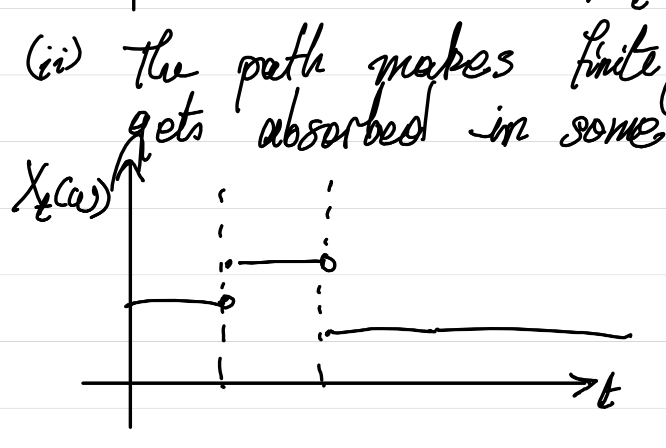


For every $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ of a right-cont. process remains constant for a while.
So there are 3 possibilities for such a path.

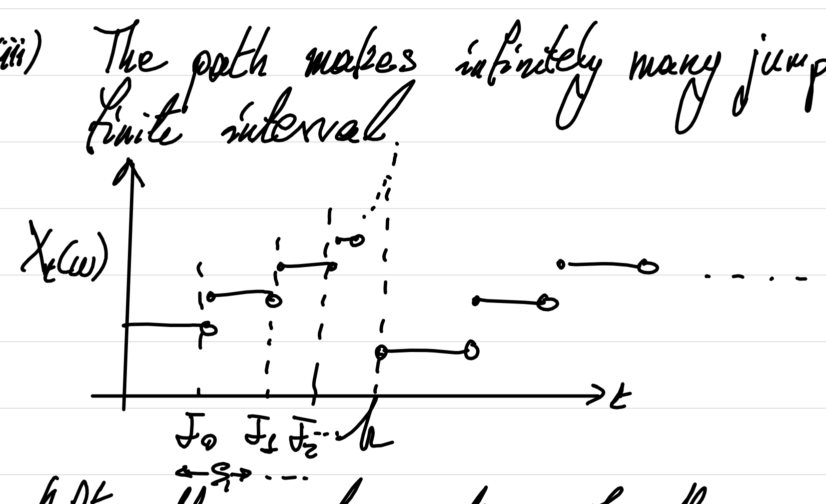
(i) The path makes infinitely many jumps but only finitely many in any finite interval.



(ii) The path makes finitely many jumps and gets absorbed in some state



(iii) The path makes infinitely many jumps in a finite interval.



After the explosion time ζ the process starts again.

Jump times $J_0, J_1, J_2, \dots$
Holding times: $S_1, S_2, S_3, \dots$

$J_0 = 0$, $J_{n+1} = \inf\{t \geq J_n : X_t \neq X_{J_n}\}$ $\forall n \geq 0$.
where $\inf \emptyset := \infty$.

And $S_n = \begin{cases} J_n - J_{n-1}, & \text{if } J_{n-1} < \infty \\ \infty, & \text{otherwise.} \end{cases}$

By right-cont., $S_n > 0$ $\forall n$.

If $J_{n+1} = \infty$ for some n, then set $X_\infty := X_{J_n}$ (finite)

The (first) explosion time ζ is defined by
$$\zeta = \sup_n J_n = \sum_{n=1}^{\infty} S_n.$$

We define the jump chain (Y_n) by setting $Y_n = X_{J_n}$ $\forall n$.

We are not going to consider what happens to a chain after first explosion. We thus set $X_t = \infty$ $\forall t \geq \zeta$.

We call such chains minimal.

Def: A cont.-time, right-cont. MC $X_t = (X_t)_{t\geq 0}$ has the Markov Property (& is called a MP) if $\forall i_1, \dots, i_n \in I$,
$$\mathbb{P}(X_{t_n} = i_n \mid X_{t_{n-1}} = i_{n-1}, \dots, X_{t_1} = i_1) = \mathbb{P}(X_{t_n} = i_n \mid X_{t_{n-1}} = i_{n-1}).$$

Remark: $\forall h > 0$, $Z_n = X(hn)$ defines a discrete-time MC. In particular, $X(1), \dots, X(n), \dots$ is a discrete-time MC.

Def: The transition probabilities are
$$p_{ij}(s,t) = \mathbb{P}(X_t = j \mid X_s = i)$$
$\forall s \leq t$, $i, j \in I$.

It is called time-homogeneous if $p_{ij}(s,t)$ depends only on $t-s$, i.e.
$$p_{ij}(s,t) = p_{ij}(0, t-s) = p_{ij}(t-s).$$

As in the discrete-time Markov Chain, a time-homogeneous MP is characterised by
(I) Its initial distribution $\lambda_i := \mathbb{P}(X_0 = i)$ $\forall i \in I$.
(II) Its transition matrices $(P(t))_{t\geq 0} = (p_{ij}(t))_{i,j\in I, t \geq 0}$.
The family $(P(t))_{t\geq 0}$ is called the transition semi-group of the MC.
It satisfies —
(I) $P(0) = Id$ (identity matrix).
(II) $P(t)$ is a stochastic matrix $\forall t \geq 0$.
(III) $P(t+s) = P(t) \cdot P(s)$ $\forall s, t \geq 0$ (Chapman-Kolm.).

Proof: (III): $\forall x, z$
$$\begin{aligned} P(t+s)_{xz} &= \mathbb{P}(X_{t+s} = z \mid X_0 = x) \\ &= \sum_y \mathbb{P}(X_{t+s} = z \mid X_t = y) \cdot \mathbb{P}(X_t = y \mid X_0 = x) \\ &= P(s)_{yz} \cdot P(t)_{xy} \\ &= (P(t) \cdot P(s))_{xz}. \end{aligned}$$

APPLIED PROBABILITY
LECTURE 2

<u>Holding times:</u> let X be a <u>(rt. cont., time-hom., cont. time)</u> MC on a <u>countable state space</u> I.
Suppose it starts at x.
<u>Q:</u> How does it stay at x?
We call S_x the holding time at x. Since X is rt. cont. $S_x > 0$. Let $s,t \geq 0$

$$\mathbb{P}(S_x > t+s \mid S_x > s) = \mathbb{P}(X_u = x \ \forall u \in [0,t+s] \mid X_u = x \ \forall u \in [0,s])$$

Markov ppty

$$= \mathbb{P}(X_u = x \ \forall u \in [s,t+s] \mid X_u = x \ \forall u \in [0,s])$$
$$= \mathbb{P}(X_u = x \ \forall u \in [s,t+s] \mid X_s = x).$$
$$= \mathbb{P}(X_u = x \ \forall u \in [0,t] \mid X_0 = x).$$
$$= \mathbb{P}(S_x > t).$$

T.H.

i.e. $\mathbb{P}(S_x > t+s \mid S_x > s) = \mathbb{P}(S_x > t)$

Thus, S_x has the <u>memory-less property</u>.
The same holds for the holding time of any state. From the next thm we get that S_x has the exponential distribution and we call its parameter q_x, i.e. $\mathbb{P}(S_x > t) = e^{-q_x t}$ $\forall t \geq 0$.

<u>Thm:</u> (<u>Memoryless ppty</u>) let S be a positive r.v. Then S has the memoryless ppty, i.e. $\mathbb{P}(S > t+s \mid S > s) = \mathbb{P}(S > t)$, $\forall s,t \geq 0$ iff S has the exponential distribution.

<u>Proof:</u> ($\Leftarrow$): easy.
($\Rightarrow$): Set $F(t) = \mathbb{P}(S > t)$

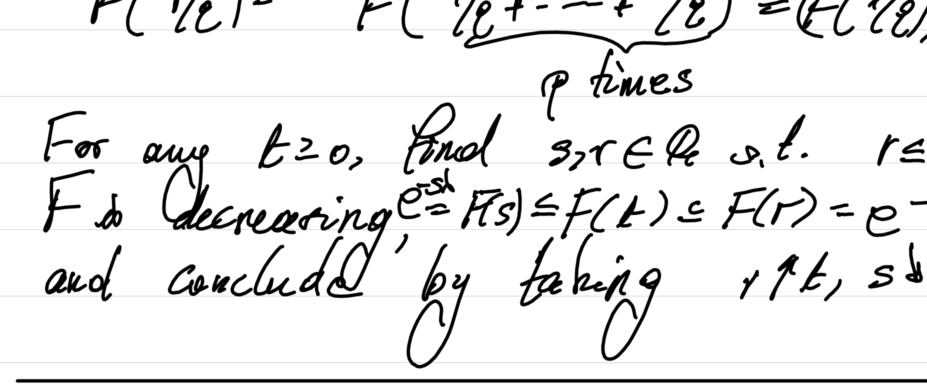
$$F(t+s) = \mathbb{P}(S > t+s) = F(t) \cdot F(s) \ \forall s,t \geq 0$$

Since $S > 0$, $\exists n \in \mathbb{N}$ s.t. $F(1/n) = \mathbb{P}(S > \frac{1}{n}) > 0$.
Then $F(1) = F(\frac{1}{n} + \cdots + \frac{1}{n}) = (F(\frac{1}{n}))^n > 0$
Hence we can set $F(1) = e^{-\lambda}$ for some $\lambda \geq 0$.
Then, $\forall k \in \mathbb{N}$ $F(k) = (F(1))^k = e^{-k\lambda}$.
$\forall p,q \in \mathbb{N}$ $(F(1/q))^q = F(1) = e^{-\lambda}$ so,
$F(1/q) = e^{-\lambda/q}$. So

$$F(p/q) = F(\underbrace{1/q + \cdots + 1/q}_{p \text{ times}}) = (F(1/q))^p = e^{-\frac{p}{q}\lambda}.$$

For any $t \geq 0$, find $s, r \in \mathbb{Q}$ s.t. $r \leq t \leq s$. As F is decreasing, $e^{-s\lambda} = F(s) \leq F(t) \leq F(r) = e^{-r\lambda}$ and conclude by taking $r \uparrow t$, $s \downarrow t \Rightarrow F(t) = e^{-\lambda t}$.

<u>Poisson process:</u> We are now going to look at the simplest (<u>and most important</u>) example of a continuous-time MC, the Poisson process on $\mathbb{R}^+$.



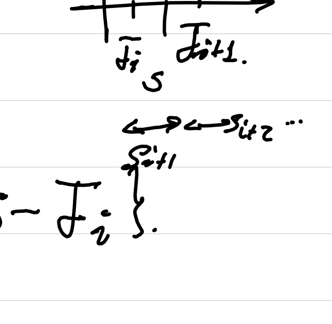
S_i iid $Exp(\lambda)$, $\lambda > 0$.

Suppose that $S_1, S_2, \ldots$, are iid with $S_i \sim Exp(\lambda)$ (i.e. $\mathbb{P}(S_1 > t) = e^{-\lambda t}$ $\forall t \geq 0$).
Define the jump times $J_0 = 0$, $J_1 = S_1, \ldots, J_n = \sum_{i=1}^{n} S_i$
and set $X_t = i$ if $J_i \leq t < J_{i+1}$ $\forall i = 0,1,2,\ldots$
Then X is called a Poisson process on $\mathbb{R}^+$ with parameter/rate λ. (P.P.(λ)). ($I = \{0,1,2,\ldots\}$)

Note that X is right-cont. and is incr.
We sometimes refer to the jumps $(J_i)_i$ as <u>points</u> of a P.P. and X_t is the # of points in $[0,t]$.

<u>Thm: (Markov PPTY):</u> let $(X_t)_{t\geq 0}$ be a P.P.(λ).
Then $\forall s \geq 0$, the process $(X_{s+t} - X_s)_{t \geq 0}$ is also a P.P.(λ) and is independent of $(X_r)_{r \leq s}$.

<u>Proof:</u> Set $Y_t = X_{s+t} - X_s$, $t \geq 0$. Let $i \in \mathbb{N} \cup \{0\}$ and condition on $\{X_s = i\}$. Then, the jump times for the process Y are
$J_{i+1} - s$, $J_{i+2} - s, \ldots$ and the holding times are $T_1 = J_{i+1} - s = J_i + S_{i+1} - s = S_{i+1} - (s - J_i)$.
$j \geq 2$ $T_j = S_{i+j}$
where J_is are the jump times and holding times for X.



Since $\{X_s = i\} = \{J_i \leq s\} \cap \{S_{i+1} > s - J_i\}$.

Conditional on $\{X_s = i\}$,

$$\mathbb{P}(T_1 > t \mid X_s = i) = \mathbb{P}(S_{i+1} > s - J_i + t \mid J_i \leq s, S_{i+1} > s - J_i)$$

conditioning on $J_i \leq s$, + indep.

$$= \mathbb{P}(S_{i+1} > t) \leftarrow \text{memoryless ppty.}$$

So, conditional on $\{X_s = i\}$, by the memoryless ppty of Exp, and $S_{i+1} \perp\!\!\!\perp J_i$, we see $T_1 \sim Exp(\lambda)$.

Moreover, the times T_j, $j \geq 2$ are indep. of S_k, $k \leq i+1$ and hence independent of $(X_r)_{r \leq s}$ and hence $Exp(\lambda)$ dist. $\square$.

Similar to this proof, one can show the <u>Strong Markov Property</u> for the P.P. Recall a r.v. T with values in $[0,\infty]$ is called a stopping time if $\forall t$, the event $\{T \leq t\}$ depends only on $(X_s)_{s \leq t}$.

<u>Thm (Strong Markov Property):</u> let $(X_t)_{t \geq 0}$ be a P.P.(λ) and T a stopping time. Then, conditioned on $\{T < \infty\}$, the process $(X_{s+T} - X_T)_{s \geq 0}$ is a P.P.(λ) $\perp\!\!\!\perp$ $(X_s)_{s \leq T}$.

The following thm gives 3 equivalent characterisations of P.P.

<u>Thm:</u> let $(X_t)_{t \geq 0}$ be an increasing process taking values in $\{0,1,2,\ldots\}$ with $X_0 = 0$. Let $\lambda > 0$. Then, TFAE
(a) The holding times $S_1, S_2, \ldots$ are iid $\sim Exp(\lambda)$ and jump chain is $Y_n = n$.

(b) (Infinitesimal defn): X has independent increments and as $h \downarrow 0$, uniformly in t,

$$\mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h)$$
$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h)$$

(c) X has indep. and stationary increments and $X_t \sim Poi(\lambda t)$ $\forall t \geq 0$, i.e.,

$$\mathbb{P}(X_t = k) = \frac{e^{-\lambda t} \lambda^k}{k!}, \ k \geq 0.$$

APPLIED PROBABILITY
LECTURE 3

Recall (Markov Property): let (X_t) be a P.P.(λ). Then $\forall s \geq 0$ the process $(X_{t+s} - X_s)_{t \geq 0}$ is also a P.P.(λ) and independent of $(X_u)_{u \leq s}$.

Thm: let X_t be an increasing right-continuous processes taking values in $\{0, 1, 2, \ldots\}$ with $X_0 = 0$. Let $\lambda > 0$. TFAE —

(a) The holding times $S_1, S_2, \ldots$ are iid $Exp(\lambda)$ and the jump chain is $Y_n = n$ $\forall n = 1, 2, \ldots$

(b) (infinitesimal): X has indep. increments & as $h \downarrow 0$, unif. in t.

$$\mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h)$$
$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h)$$

(c) X has indep. increments and is stationary incs. and $\forall t > 0$, $X_t \sim \mathcal{P}(\lambda t)$.

i.e. $\mathbb{P}(X_t = k) = \dfrac{(\lambda t)^k e^{-\lambda t}}{k!}$, $k \geq 0$.

Proof: (a) $\Rightarrow$ (b). If (a) holds, by the Markov Property, the increments are indep. and stationary. So uniformly in t, as $h \downarrow 0$,

$$\mathbb{P}(X_{t+h} - X_t = 0) \overset{\text{stationarity}}{=} \mathbb{P}(X_h = 0) = \mathbb{P}(S_1 \geq h) = e^{-\lambda h} = 1 - \lambda h + o(h).$$

$$\mathbb{P}(X_{t+h} - X_t = 1) = \mathbb{P}(X_h = 1) = \mathbb{P}(S_1 \leq h) = \lambda h + o(h).$$

$$\mathbb{P}(X_{t+h} - X_t \geq 2) = \mathbb{P}(X_h \geq 2) = \mathbb{P}(S_1 + S_2 \leq h) \leq \mathbb{P}(S_1 \leq h, S_2 \leq h) = o(h)$$

$$\Rightarrow \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h).$$

(b) $\Rightarrow$ (c): If X satisfies (b), then $(X_{s+t} - X_s)_{t \geq 0}$ also satisfies (b) and hence X has indep. and stationary increments.

It remains to show that $X_t \sim \mathcal{P}oi(\lambda t)$.
Fix $t > 0$. Set $p_j(t) = \mathbb{P}(X_t = j)$.
Since the increments of X are indep. and increasing, $\forall j \geq 1$, we get

$$p_j(t+h) = \mathbb{P}(X_{t+h} = j) \overset{\text{ind. inc.}}{=} \sum_{i=0}^{j} \mathbb{P}(X_t = j - i)\mathbb{P}(X_{t+h} - X_h = i)$$
$$= \mathbb{P}(X_t = j) \cdot \mathbb{P}(X_{t+h} - X_h = 0) + \mathbb{P}(X_t = j-1) \cdot \mathbb{P}(X_{t+h} - X_h = 1) + o(h)$$

i.e. $p_j(t+h) = p_j(t)(1 - \lambda h + o(h)) + p_{j-1}(t)(\lambda h + o(h)) + o(h)$

$$= p_j(t)(1 - \lambda h) + p_{j-1}(t)\lambda h + o(h)$$

$$\frac{p_j(t+h) - p_j(t)}{h} = -\lambda p_j(t) + \lambda p_{j-1}(t) + o(1)$$

As $h \downarrow 0$, $p_j'(t) = -\lambda p_j(t) + \lambda p_{j-1}(t)$

Differentiating $e^{\lambda t} p_j(t)$ wrt, we get

$$(e^{\lambda t} p_j(t))' = (\lambda p_j(t) + p_j'(t))e^{\lambda t}$$
$$= \lambda p_{j-1}(t) e^{\lambda t}, \quad j \geq 1. \ (*)$$

For $j = 0$, $p_0(t+h) = p_0(t) \cdot \mathbb{P}(X_{t+h} - X_t = 0) = p_0(t) \cdot (1 - \lambda t + o(h))$

i.e., $\dfrac{p_0(t+h) - p_0(t)}{h} = -\lambda p_0(t) + o(1)$

As $h \downarrow 0$, $p_0'(t) = -\lambda p_0(t)$
i.e. $p_0(t) = e^{-\lambda t}$ (as $p_0(0) = 1$).

From $(*)$ with $j = 1$, $(e^{\lambda t} p_1(t))' = \lambda \underbrace{e^{\lambda t} p_0(t)}_{=1} = \lambda$

$$\Rightarrow p_1(t) = \lambda e^{-\lambda t}, \ t \geq 0.$$

By induction, $p_j(t) = e^{-\lambda t} \dfrac{(\lambda t)^j}{j!}$.

(c) $\Rightarrow$ (a): Observe that (c) determines the FDD of X. As X is rt-conti, then (c) determines X. Already have shown that (a) $\Rightarrow$ (c). Thus, (c) $\Rightarrow$ (a) □

((a) $\Rightarrow$ (c) directly)

$$\mathbb{P}(X_t = n) = \mathbb{P}(S_1 + S_2 + \cdots + S_n \leq t < S_1 + \cdots + S_{n+1})$$
$$= \mathbb{P}(\underbrace{S_1 + \cdots + S_n}_{\Gamma(n,\lambda)} \leq t) - \mathbb{P}(\underbrace{S_1 + \cdots + S_{n+1}}_{\Gamma(n+1,\lambda)} \leq t)$$

$S_i \overset{iid}{\sim} Exp(\lambda)$

$$= \int_0^t \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} dx - \int_0^t \lambda e^{-\lambda x} \frac{(\lambda x)^n}{n!} dx$$
$$\overset{?}{=} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad \text{[Integrate by parts.}$$

Thm: (Superposition): Let X and Y be indep. Poisson processes with rates λ and μ resp. Then $Z_t = X_t + Y_t$ is a P.P.$(\lambda + \mu)$.

Proof: by (c): Stationary and indep. inc & $X_t + Y_t \sim \mathcal{P}oi(\lambda + \mu)$.

by (b): $\mathbb{P}(Z_{t+h} - Z_t = 1)$
$= \mathbb{P}(X_{t+h} - X_t = 1) \cdot \mathbb{P}(Y_{t+h} - Y_t = 0) + \mathbb{P}(X_{t+h} - X_t = 0) \cdot \mathbb{P}(Y_{t+h} - Y_t = 1)$
$= (\lambda + \mu)h + o(h)$ uniformly in $t \geq 0$

APPLIED PROBABILITY
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<u>Thm</u>: (Thinning/Colouring) Let X be $PP(\lambda)$ and let $(Z_i)_i$ be iid Ber r.v.'s with success prob. p. Let Y be a process with values $\mathbb{N}$, which jumps at time t. If X jumps at t and $Z_{X_t}=1$.
In other words, we keep every point with probability p indep. over different points.
Then $Y \sim PP(p\lambda)$ and $X-Y \sim PP((1-p)\lambda)$ and $Y \perp\!\!\!\perp X-Y$.

<u>Proof</u>: We shall use the infinitesimal definition. The indep. of increments of Y follows from that of X.

$$\mathbb{P}(Y_{t+h}-Y_t=1) = p\cdot\mathbb{P}(X_{t+h}-X_t=1) = p\lambda h + o(h).$$

$$\begin{aligned}\mathbb{P}(Y_{t+h}-Y_t=0) &= \mathbb{P}(X_{t+h}-X_t=0) + (1-p)\cdot\mathbb{P}(X_{t+h}-X_t=1)+o(h)\\ &= 1-\lambda h+(1-p)\lambda h+o(h)\\ &= 1-p\lambda h+o(h).\end{aligned}$$

Thus, $Y \sim PP(p\lambda)$.
Similarly, $X-Y \sim PP((1-p)\lambda)$.

To prove the indep. since both processes are right-continuous and increasing, enough to check that the f.d.d. are indep. i.e. for $t_1<t_2<\dots<t_k$, $m_1\le m_2\le\dots\le m_k$ & $n_1\le n_2\le\dots\le n_k$

$$\mathbb{P}(Y_{t_1}=n_1,\dots,Y_{t_k}=n_k, X_{t_1}-Y_{t_1}=m_1,\dots,X_{t_k}-Y_{t_k}=m_k)$$
NTS
$$= \mathbb{P}(Y_{t_1}=n_1,\dots,Y_{t_k}=n_k)\times\mathbb{P}(\ \ )$$

Will only show $k=1$. General case follows similarly.

$$\begin{aligned}&\mathbb{P}(Y_t=n, X_t-Y_t=m)\\ &= \mathbb{P}(Y_t=n, X_t=n+m)\\ &= \mathbb{P}(Y_t=n \mid X_t=n+m)\,\mathbb{P}(X_t=n+m) \quad [\mathbb{P}(X_t=n+m): P_0(\lambda t)]\\ &= \binom{m+n}{n}p^n(1-p)^m\,\frac{(\lambda t)^{n+m}e^{-\lambda t}}{(n+m)!} \quad [\mathrm{Bin}(m+n,p)]\\ &= \frac{(p\lambda t)^n e^{-p\lambda t}}{n!}\,\frac{((1-p)\lambda t)^m e^{-(1-p)\lambda t}}{m!}\\ &= \mathbb{P}(Y_t=n)\times\mathbb{P}(X_t-Y_t=m).\end{aligned}$$

<u>Thm</u>: Let X be a P.P.(λ). Conditional on the event $\{X_t=n\}$, the jump times $J_1, J_2,\dots,J_n$ have joint density

$$f(t_1,\dots,t_n)=\frac{n!}{t^n}\mathbb{1}(0\le t_1\le t_2\le\dots\le t_n\le t)$$

i.e., that of the order statistics of n iid Unif$[0,t]$ r.v.s.

<u>Proof</u>: Since, $S_1, S_2,\dots$ are iid $\mathrm{Exp}(\lambda)$, the joint density $(S_1,\dots,S_{n+1})$ is

$$\lambda^{n+1}e^{-\lambda(s_1+\dots+s_{n+1})}\mathbb{1}(s_1,\dots,s_{n+1}\ge 0).$$

Thus, the jump times $J_1=S_1$, $J_2=S_1+S_2$, ..., $J_{n+1}=S_1+\dots+S_{n+1}$ have the joint density (Jacobian transformation has det 1)

$$g(t_1,\dots,t_{n+1})=\lambda^{n+1}e^{-\lambda t_{n+1}}\mathbb{1}(0\le t_1\le t_2\le\dots\le t_{n+1}).$$

Now take $A\subseteq\mathbb{R}^n$, then

$$\begin{aligned}&\mathbb{P}((J_1,\dots,J_n)\in A, X_t=n)\\ &= \mathbb{P}((J_1,\dots,J_n)\in A, J_n\le t<J_{n+1})\\ &= \int_{(t_1,\dots,t_n)\in A,\ t_n\le t<t_{n+1}}\lambda^{n+1}e^{-\lambda t_{n+1}}\mathbb{1}(0\le t_1\le\dots\le t_{n+1})\,dt_1\cdots dt_n\,dt_{n+1}\\ &= \int_{(t_1,\dots,t_n)\in A}\lambda^{n+1}\mathbb{1}(0\le t_1\le\dots\le t_n)\int_t^\infty e^{-\lambda t_{n+1}}dt_{n+1}\,dt_1\cdots dt_n\\ &= \int_{(t_1,\dots,t_n)\in A}\lambda^n e^{-\lambda t}\mathbb{1}(0\le t_1\le\dots\le t_n\le t)\,dt_1\cdots dt_n.\end{aligned}$$

Dividing by $\mathbb{P}(X_t=n)=e^{-\lambda t}\frac{(\lambda t)^n}{n!}$, we get,

$$\mathbb{P}((J_1,\dots,J_n)\in A\mid X_t=n)=\int_{(t_1,\dots,t_n)\in A}\frac{n!}{t^n}\mathbb{1}(0\le t_1\le\dots\le t_n\le t)\,dt_1\cdots dt_n$$

$\square$

<u>Birth Process</u>: A generalisation of a P.P.
For the birth process, for each i, let $S_i\sim\mathrm{Exp}(q_i)$, $i=1,2,3,\dots$ and $S_1, S_2,\dots$ are indep. Set $J_i=S_1+\dots+S_i$, and

$$X_t=i \text{ if } J_i\le t<J_{i+1}.$$

<u>Examples</u>: (0) P.P.(λ) where $q_i=\lambda\ \forall i$.
(1) <u>Simple Birth Process</u>: where $q_i=i\lambda$ for some $\lambda>0$, $i=1,2,\dots$ (with parameter λ)

<u>Prop (§)</u>: Let $(T_k)_{k\ge1}$ be a sequence of ind. r.v.s with $T_k\sim\mathrm{Exp}(q_k)$ and $0<q=\sum_k q_k<\infty$. Then set $T=\inf_k T_k$. Then

(a) $T\sim\mathrm{Exp}(q)$
(b) the infimum is attained at a unique K a.s.
(c) T & K are indep.

with $\mathbb{P}(K=k)=\frac{q_k}{q}$.

<u>Proof</u>: ES 1.

In a population, each individual gives birth after an indep. Exp(λ) time. If we have i individuals, the first birth happens at Exp$(i\lambda)$ time, and by the memoryless ppty, the process begins afresh. This is the <u>Simple Birth Process</u>

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The main difference between P.P. and a Birth Process is the possibility of explosion.

Recall the explosion time $\zeta = \sup_n J_n = \sum_n S_n$.

We say an explosion happens if $\zeta < \infty$.

<u>Prop:</u> let X be a birth process, with rates (q_i), with $X_0 = 1$.
(1) if $\sum_i \frac{1}{q_i} < \infty$, then $\mathbb{P}(\zeta < \infty) = 1 \to$ exp. w.p. 1
(2) if $\sum_i \frac{1}{q_i} = \infty$, then $\mathbb{P}(\zeta = \infty) = 1 \to$ n. exp. w.p. 1.

<u>Eg:</u> P.P.(λ) is non-explosive. Simple-Birth process is non-explosive.

<u>Thm (Monotone convergence thm):</u> If $X_n \geq 0$ $\forall n$, $X_n \uparrow X$, then $\mathbb{E}X_n \uparrow \mathbb{E}X$.

<u>Cor 1:</u> If $Y_n \geq 0$, then $\mathbb{E}[\sum_n Y_n] = \sum_n \mathbb{E}[Y_n]$

<u>Cor 2:</u> If $Y_n \leq M$, and $Y_n \downarrow Y$, then $\mathbb{E}Y_n \downarrow \mathbb{E}Y$.

<u>Proof of Prop:</u> (1) $\mathbb{E}[\sum_n S_n] \overset{(MCT)}{=} \sum_n \mathbb{E}[S_n] = \sum_n \frac{1}{q_n} < \infty$, ($S_n \sim Exp(q_n)$)

So $0 \leq \zeta = \sum_n S_n < \infty$ a.s., $\mathbb{P}(\zeta < \infty) = 1$.

(2) $$\mathbb{E}[e^{-\sum_{n=1}^{\infty} S_n}] \overset{(MCT)}{=} \lim_n \mathbb{E}[e^{-\sum_{i=1}^{n} S_n}].$$

$$= \lim_n \prod_{i=1}^{n} \mathbb{E}[e^{-S_i}] \text{ as } S_i \text{ iid.}$$

$S_i \sim Exp(q_i)$
$$= \lim_n \prod_{i=1}^{n} \frac{1}{1 + 1/q_i} = \frac{1}{\prod_{i=1}^{\infty}(1 + \frac{1}{q_i})}$$
$$\leq \frac{1}{1 + \sum_{i=1}^{\infty} \frac{1}{q_i}} = 0.$$

Thus, $e^{-\sum S_i} = 0$ a.s., i.e. $\sum S_i = \infty$ a.s. $\mathbb{P}(\zeta = \infty) = 1$. □

<u>Thm (Markov Property):</u> let X be a Birth Process $(q_i)_i$. Then conditional on $\{X_s = i\}$, the process $(X_{s+t})_{t \geq 0}$ will be a birth process with rates $(q_j)_{j \geq i}$ and is indep. of $(X_r)_{r \leq s}$.

<u>Thm:</u> Let X be an increasing right-cont. process with values in $\{1, 2, \ldots\} \cup \{\infty\}$. let $0 \leq q_j < \infty$ $\forall j$. Then TFAE —

(I) (Jump chain/holding time def): Conditional on $X_0 = i$ the holding times $S_1, S_2, \ldots$ are indep. Exp with rates $q_i, q_{i+1}, \ldots$ and the jump chain is given by $Y_n = i + n$, $\forall n = 1, 2, \ldots$.
(II) (Infinitesimal def): $\forall t, h > 0$, conditional on $\{X_t = i\}$, the process (X_{t+h}) is indep. of $(X_s)_{s \leq t}$ and as $h \downarrow 0$, uniformly in t,
$\mathbb{P}(X_{t+h} = i \mid X_t = i) = 1 - q_i h + o(h)$
$\mathbb{P}(X_{t+h} = i+1 \mid X_t = i) = q_i h + o(h)$
(III) (Transition probability defn): $\forall n = 0, 1, \ldots$, and all times $0 \leq t_0 \leq t_1 \leq \ldots \leq t_{n+1}$ and all states $i_0, i_1, \ldots, i_{n+1}$.
$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_{1:n}} = i_{1:n}) = \mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_n} = i_n)$$
$$= \mathbb{P}(X_{t_{n+1} - t_n} = i_{n+1} \mid X_0 = i_n).$$
$$:= p_{i_n i_{n+1}}(t_{n+1} - t_n)$$

where $(p_{ij}(t))$ is the <u>unique</u> solution to the (forward) eqns —

$$p_{ij}'(t) = q_{j-1} p_{i,j-1}(t) - q_j p_{i,j}(t) \quad (*)$$

$$\left(\begin{array}{l} \text{As in P.P.} \\ p_{ij}(t+h) = p_{i,j-1}(t) \cdot q_{j-1} \cdot h + p_{ij}(t)(1 - q_j h) + o(h) \end{array} \right)$$

<u>Existence and Uniqueness:</u>
For $j = 1$, $p_{i,i}'(t) = -q_i p_{i,i}(t)$, $p_{i,i}(0) = 1$.
So $p_{i,i}(t) = e^{-q_i t}$.
By induction, if the unique solution for $p_{ij}(t)$, then sub. it in $(*)$ for $j+1$ to see a unique solution.

Write $P(t) = (p_{ij}(t))_{ij}$
$P'(t) = (p_{ij}'(t))_{ij}$

Then $(*)$ is $P'(t) = P(t)Q$
where $$Q = \begin{pmatrix} -q_1 & q_1 & & & \\ & -q_2 & q_2 & & 0 \\ 0 & & \ddots & \ddots & \\ & & & -q_i & q_i \\ & & & & \ddots \end{pmatrix}$$

<u>Construction of Markov Process:</u>

<u>Def:</u> $Q = (q_{ij})_{i,j \in I}$ is called a Q matrix (or generator matrix/infinitesimal generator) if —
(a) $0 \leq -q_{ii} < \infty$ $\forall i$
(b) $0 \leq q_{ij}$, $i \neq j$
(c) $\sum_j q_{ij} = 0$ $\forall i$.

Write $q_i = -q_{ii} = \sum_{j \neq i} q_{ij}$ $\forall i \in I$. Given a Q-matrix, define the jump-matrix $\tilde{P}$ as follows —
for $x \neq y$, with $q_x \neq 0$, set $\tilde{p}_{xy} = \frac{q_{xy}}{q_x}$
and $\tilde{p}_{xx} = 0$. If $q_x = 0$, set $\tilde{p}_{xy} = \mathbb{1}(x = y)$.

<u>Eg:</u> If $Q = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}$, then $\tilde{P} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$

Clearly $\tilde{P}$ is a stochastic matrix.

<u>Def:</u> let Q be a Q-matrix. Then a (minimal) random process X is a Markov process with initial distribution λ and infinitesimal generator Q if —
(a) The jump chain $Y_n = X_{J_n}$ is a discrete-time MC with $Y_0 \sim \lambda$ and transition matrix $\tilde{P}$.
(b) Conditional on $Y_0, Y_1, \ldots, Y_n$, the following times $S_1, S_2, \ldots, S_{n+1}$ are iid with $S_i \sim Exp(q_{Y_{i-1}})$ for $i = 1, 2, \ldots$.
We write $X \sim Markov(\lambda, Q)$.

<u>Example:</u> Birth processes are Markov(Q) with $I = \mathbb{N}$ and $Q = \begin{pmatrix} -q_1 & q_1 & & 0 \\ & 0 & \ddots & \\ & & -q_i & q_i & \ddots \end{pmatrix}$. Note

$\tilde{P} = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 & \ddots \end{pmatrix}$, i.e. the jump chain goes from i to $i+1$ w.p. 1.

i.e. $1 \xrightarrow{1} 2 \xrightarrow{1} 3$. I.e. $Y_n = n + Y_0$.

Constructions of a Markov(Q,λ) process:

Construction 1: • (Y_n) is a discrete-time MC, $Y_0 \sim \lambda$ and transition matrix P.
• $(T_i)_{i\geq 1}$ be iid $Exp(1)$, indep of Y and set $S_n = \frac{T_n}{q_{Y_{n-1}}}$, $J_n = \sum_{i=1}^{n} S_i$.
• Set $X_t = Y_n$ in $t \in [J_n, J_{n+1})$ and $X_t = \infty$ otherwise

(If $T \sim Exp(1)$, then $T/\mu \sim Exp(\mu)$).

Construction 2: let • $(T_n^y)_{n\geq 1, y\in I}$ be iid $Exp(1)$.
• $Y_0 \sim \lambda$, and inductively define Y_n, S_n as follows
if $Y_n = x$, then for $y \neq x$,

$$S_{n+1}^y = \frac{T_{n+1}}{q_{xy}} \sim Exp(q_{xy}) \text{ and}$$

$$S_{n+1} = \inf_{y\neq x} S_{n+1}^y \sim Exp\Big(\underbrace{\sum_{y\neq x} q_{xy}}_{q_x}\Big)$$

Also, by Prop 3, $S_{n+1} = S_{n+1}^Z$ for some random Z and take $Y_{n+1} = Z$ if $q_x > 0$
$Y_{n+1} = x$ if $q_x = 0$.

(i.e., the rate at which we leave x is q_x and rate of transition from x to y is q_{xy}).

Construction 3: For $x \neq y$, let $(N_t^{x,y})$ be iid Poisson processes with rates q_{xy}. Let $Y_0 \sim \lambda$ and $J_0 = 0$, and define inductively,

$$J_{n+1} = \inf\{t \geq J_n : N_t^{Y_n,y} \neq N_{J_n}^{Y_n,y} \text{ for some } y \neq Y_n\}$$

$$Y_{n+1} = \begin{cases} y & \text{if } J_{n+1} < \infty \text{ and } N_{J_{n+1}}^{Y_n,y} \neq N_{J_n}^{Y_n,y} \\ x & \text{if } J_{n+1} = \infty \end{cases}$$

Check that this is the same as Construction 2.

For both processes, we characterized when non-explosion happens. In general, the next thm gives us some sufficient conditions — (Recall $\zeta = \sup_n J_n = \sum_n S_n$).

Thm: let $X \sim$ Markov(Q,λ) on I. Then $P(\zeta = \infty) = 1$ (i.e. no explosion), if any of the following holds
(a) I is finite
(b) $\sup_{x\in I} q_x < \infty$
(c) $X_0 = x$ and x is recurrent for the jump chain.

Since (a) $\Rightarrow$ (b), enough to show for (b).
Assume (b) holds. Set $q = \sup_x q_x < \infty$.
Then for the holding times S_n, we have $S_n = \frac{T_n}{q_{Y_{n-1}}} \geq \frac{T_n}{q}$, T_n iid $Exp(1)$
then, $\sum_n S_n \geq \frac{1}{q}\left(\sum T_n\right) = \infty$ a.s. by SLLN,
i.e. $P(\zeta = \infty) = 1$.

Assume (c) holds: let $(N_i)_{i\geq 1}$ be the times when the jump chain Y visits x. Then $(N_i)_{i\geq 1}$ is an infinite seq. w.p. 1 by the recurrence prop.
Thus, $\zeta = \sum S_n \geq \sum_{i=1}^{\infty} S_{N_i+1} = \sum_{i=1}^{\infty} \frac{T_{N_i+1}}{q_x}$

$$= \frac{1}{q_x} \sum_{i=1}^{\infty} T_{N_i+1} = \infty$$

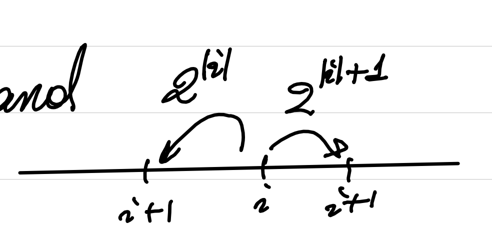
$q_x = q_{Y_{N_i}}$, iid, by SLLN.

Examples: $I = \mathbb{Z}$

i.e. $q_{i,i+1} = q_{i,i-1} = 2^{|i|}$ $\forall i \in \mathbb{Z}$,
$q_{i,i} = -2\cdot 2^{|i|}$ & $q_{ij} = 0$ otherwise

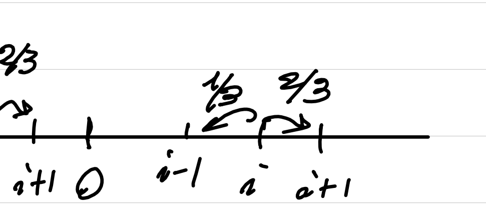
Is the chain explosive?

The jump chain Y is



So Y is a simple symmetric random walk on $\mathbb{Z}$ which is recurrent & irreducible.
Hence by Thm (c), X is non-explosive w.p. 1.

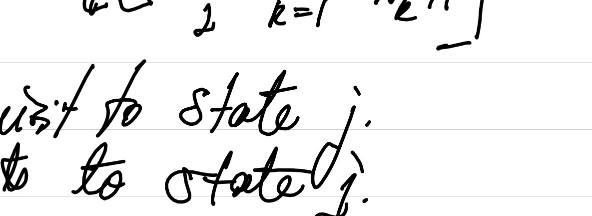
Example 2: $I = \mathbb{Z}$ and



$q_{i,i+1} = 2^{|i|+1}$, $q_{i,i-1} = 2^{|i|}$
$q_{i,i} = -3\cdot 2^{|i|}$ and $q_{i,j} = 0$ otherwise

Is X explosive?

The jump chain Y is a biased SRW on $\mathbb{Z}$, so transient



$$E[\zeta] = E\Big[\sum_n S_n\Big] = E\Big[\sum_j \sum_{k=1}^{V_j} S_{N_k^j+1}\Big]$$

N_k^j is the k^{th} visit to state j.
V_j total # visits to state j.
By transience, $V_j < \infty$ a.s.

$$\overset{MCT}{=} \sum_{j\in\mathbb{Z}} E\Big[\sum_{k=1}^{V_j} \frac{T_{N_k^j+1}}{q_j}\Big] \quad \text{iid}$$

$$= \sum_{j\in\mathbb{Z}} \frac{1}{q_j} E[V_j]\cdot \underset{\overset{\|}{1}}{E[T_1]} \text{ by independence.}$$

visits to j starting from x
$\leq 1 + \#\{$visits to j starting from $j\}$
So $E[V_j] \leq 1 + E_j V_j = 1 + E_0[V_0]$ (by symmetry)
$\leq C$.

$$\leq C \sum_{j\in\mathbb{Z}} \frac{1}{q_j} = C \sum_{j\in\mathbb{Z}} \frac{3}{2^{|j|}} < \infty.$$

So $E[\zeta] < \infty$, so $P(\zeta < \infty) = 1$ (i.e. explosive w.p. 1).

Thm: (Strong Markov Property): Let X be Markov(Q, λ), and T a stopping time. Then conditional on $T < \zeta$ and $X_T = x$, the process $(X_{T+t})_{t \geq 0}$ is Markov(Q, δ_x) (where $\delta_x(x) = 1$, $\delta_x(y) = 0$ $y \neq x$) and is indep. of $(X_s)_{s \leq T}$.

Proof: Skipped (See James Norris (6.5))

Kolmogorov's forward and backward eqn's.

Thm: Let X be a minimal right-continuous process with values in a countable set I. Let Q be a Q-matrix. Then TFAE

(a) X is a cont. time MC with generator Q.

(b) $\forall n \geq 0$, $0 \leq t_0 \leq \dots \leq t_{n+1}$ and all states $x_0, \dots, x_{n+1} \in I$,
$$\mathbb{P}(X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n, \dots, X_{t_0} = x_0) = p_{x_n, x_{n+1}}(t_{n+1} - t_n),$$
where $(P(t)) = (p_{xy}(t))_{x,y \in I}$ is the minimal non-negative solution to the backward equation: (✱) $P'(t) = QP(t)$ and $P(0) = I$.
(minimality means: if $(\bar{p}_{xy}(t))$ is another non-negative solution then $p_{xy}(t) \leq \bar{p}_{xy}(t)$ $\forall t, \forall x, y$).
In fact, if X is non-explosive, the solution is unique.

(c) $P(t)$ is also the minimal non-neg. solution to the following eq:
$$P'(t) = P(t)Q, \quad \text{and } P(0) = I.$$

(Will prove (a) $\Leftrightarrow$ (b). The equivalence of (a) $\Leftrightarrow$ (c) is similar, but requires an extra "reversibility lemma", see Norris Thm 2.8.6)

Proof: (a) $\Rightarrow$ (b): If (J_n) denotes the jump times, we have,
$$\mathbb{P}_x(X_t = y, t < J_1) = \mathbb{1}(x = y)\, e^{-q_x t}$$
For $z \neq x$
$$\mathbb{P}_x(X_t = y,\ J_1 \leq t,\ X_{J_1} = z)$$
$$= \int_0^t q_x e^{-q_x s} \frac{q_{xz}}{q_x} \cdot p_{zy}(t-s)\, ds$$
Indep. of J_1 and jump chain
$$= \int_0^t e^{-q_x s} q_{xz} \cdot p_{z,y}(t-s)\, ds$$
summing up
$$\mathbb{P}(X_t = y) = e^{-q_x t} \mathbb{1}(x = y) + \sum_{z \neq x} \int_0^t e^{-q_x s} q_{xz}\, p_{zy}(t-s)\, ds$$
$$= e^{-q_x t} \mathbb{1}(x = y) + \int_0^t \sum_{z \neq x} p_{zy}(t-s) e^{-q_x s} q_{xz}\, ds$$
or,
$$e^{q_x t} p_{xy}(t) = \mathbb{1}(x = y) + \int_0^t \sum_{z \neq x} e^{q_x u} q_{xz} p_{zy}(u)\, du$$
$$\sum_{z \neq x} \underbrace{e^{q_x u}}_{\leq e^{q_x t}} \underbrace{q_{xz}}_{\sum_{z \neq x} \cdot = q_x} \underbrace{p_{zy}(u)}_{\leq 1} \leq q_x e^{q_x t}.$$

In particular, $p_{xy}(t)$ is cont. in t.
So $\sum_{z \neq x} e^{q_x u} q_{xz} p_{zy}(u)$ is a uniformly conv. series of continuous functions, which is continuous, hence RHS is differentiable.

Differentiating,
$$\cancel{e^{q_x t}}(p_{xy}'(t) + q_x p_{xy}(t)) = \sum_{z \neq x} \cancel{e^{q_x t}} q_{xz} p_{zy}(t)$$
i.e., $p_{xy}'(t) = \sum_{z} q_{xz} p_{zy}(t)$, i.e. $P'(t) = QP(t)$

Minimality: Let $\bar{P}$ be another non-negative soln. to the backward eq. Will show that $p_{xy}(t) \leq \bar{p}_{xy}(t)$ $\forall t$.

By reversing the steps, we get
$$\bar{p}_{xy}(t) = e^{-q_x t} \mathbb{1}(x = y) + \int_0^t \sum_{z \neq x} e^{-q_x s} q_{xz} \cdot \bar{p}_{zy}(t-s)\, ds$$

We will prove inductively that
$$\mathbb{P}_x(X_t = y, t < J_n) \leq \bar{p}_{xy}(t)$$

For $n = 1$, LHS $= \mathbb{1}(x = y) e^{-q_x t} \leq \bar{p}_{xy}(t)$.
Assume $\mathbb{P}_x(X_t = y, t < J_n) \leq \bar{p}_{xy}(t)$ $\forall t$. — (✱)
For $n+1$,
$$\mathbb{P}_x(X_t = y, t < J_{n+1}) = \mathbb{P}_x(X_t = y, t < J_1 < J_{n+1}) + \mathbb{P}_x(X_t = y, J_1 \leq t < J_{n+1})$$
$$= \mathbb{1}(x = y) e^{-q_x t} + \int_0^t \sum_{z \neq x} \cancel{q_x} e^{-q_x s} \frac{q_{xz}}{\cancel{q_x}} \underbrace{\mathbb{P}_z(X_{t-s} = y, t - s < J_n)}_{(✱)\ \leq \bar{p}_{zy}(t-s)} ds$$
$$\leq \bar{p}_{xy}(t).$$

So $\mathbb{P}_x(X_t = y, t < J_n) \leq \bar{p}_{xy}(t)$ $\forall n$. Let $n \to \infty$, then,
$$\mathbb{P}_x(X_t = y, t < \sup_n J_n) = \lim_{n \to \infty} \mathbb{P}_x(X_t = y, t < J_n) \leq \bar{p}_{xy}(t)$$

By minimality
$$p_{xy}(t) = \mathbb{P}_x(X_t = y, t < \zeta) \leq \bar{p}_{xy}(t).$$
$\square$

(b) $\Rightarrow$ (a): (b) determines the f.d.d. of X, and hence X as X is right-continuous.
Also (a) $\Rightarrow$ (b) Hence (b) $\Rightarrow$ (c).

Finite State space: If I is finite, the transition semi-group P has a simple expression in terms of Q.

Def: (Matrix Exponential) If A is a finite-dim square matrix, its matrix exponential is defined as
$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

Lemma: For any $r \times r$ matrix A, the exponential e^A is again $r \times r$ matrix and if A_1, A_2 commute, then
$$e^{A_1 + A_2} = e^{A_1} \cdot e^{A_2} = e^{A_2} \cdot e^{A_1}.$$

$$\left(|A_{ij}^n| \leq r^n \left(\max_{i,j} |A_{ij}|\right)^n,\ (A_1 + A_2)^n = \sum_{k=0}^{n} \binom{n}{k} A_1^k A_2^{n-k}\right)$$

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If A_1, A_2 are fin. dim. matrices, then $e^{A_1+A_2} = e^{A_1} \cdot e^{A_2}$.

I finite $\quad P'(t) = P(t)Q \iff P(t) = e^{tQ}$.

__Prop:__ Let Q be a Q-matrix on a finite set I and let $P(t) = e^{tQ}$. Then

(i) $P(t+s) = P(t)\cdot P(s)$ $\forall s,t$.

(ii) $(P(t))$ is the <u>unique soln.</u> to the following eq.: $P'(t) = P(t)Q$, $P(0) = I$.

(iii) $(P(t))$ is the unique soln. to the backward eq. $P'(t) = Q\cdot P(t)$, $P(0) = I$.

(iv) For $k = 0, 1, 2, \ldots$ $\quad Q^k = \left(\frac{d}{dt}\right)^k\Big|_{t=0} P(t)$.

__Proof:__ (i) As tQ and sQ commute.

(ii)&(iii) the sum in e^{tQ} has infinite radius of convergence so can differentiate term by term, to get,

$$\frac{d}{dt}P(t) = \frac{d}{dt}\sum_k^{\infty}\frac{(tQ)^k}{k!} = \sum_{k\geq 0}^{\infty}\frac{d}{dt}\frac{(tQ)^k}{k!}$$
$$= \sum_{k\geq 1}^{\infty}\frac{t^{k-1}Q^k}{(k-1)!}$$
$$= \sum_{k=1}^{\infty}\frac{(tQ)^{k-1}}{(k-1)!}Q = e^{tQ}Q$$

Similarly, $\frac{d}{dt}P(t) = QP(t)$.

If $\bar{P}$ is another solution to the forward equation

$$\frac{d}{dt}\left(\bar{P}(t)e^{-tQ}\right) = \bar{P}'(t)e^{-tQ} + \bar{P}(t)(-Qe^{-tQ})$$

differentiate term-by-term

$$= \bar{P}(t)Qe^{-tQ} - \bar{P}(t)Qe^{-tQ}$$
$$= 0$$

and $\bar{P}(0) = I$, so $\bar{P}(t)e^{-tQ} = I$ $\forall t \geq 0$, $\sum\frac{(tQ)^k}{k!}$

i.e. $\bar{P}(t) = e^{tQ}$.

(iv) Differentiate term-by-term.

__Example:__ Let $Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}$.

Find $p_{11}(t)$.

Diagonalise Q: Eigenvalues are $0, -2, -4$.

So $Q = UDU^{-1}$, where $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}$

thus $$e^{tQ} = \sum_{k=0}^{\infty}\frac{(tQ)^k}{k!}$$
$$= U\sum_{k=0}^{\infty}\frac{t^kD^k}{k!}U^{-1} = U\begin{pmatrix} \sum\frac{0^k}{k!} & & \\ & \sum\frac{(-2t)^k}{k!} & \\ & & \sum\frac{(-4t)^k}{k!}\end{pmatrix}U^{-1}$$
$$= U\cdot\mathrm{diag}(1, e^{-2t}, e^{-4t})\cdot U^{-1}.$$

i.e. $p_{11}(t) = a + be^{-2t} + ce^{-4t}$.

Have $p_{11}(0) = 1$, $p_{11}'(0) = q_{11}$, $p_{11}''(0) = q_{11}^2$.

__Prop:__ Let I be a finite state space and let Q be a matrix. Then it is a Q matrix iff $P(t) = e^{tQ}$ is a stochastic matrix $\forall t \geq 0$.

__Proof:__ For $t \downarrow 0$, $P(t) = I + tQ + O(t^2)$.

Hence $\forall x \neq y$, $q_{xy} \geq 0$ iff $p_{xy}(t) \geq 0$ $\forall t \geq 0$ suff. small

Also, $q_{xx}(t) \geq 0$ for $t \geq 0$ suff. small.

Since $P(t) = (P(t/n))^n$, we get $q_{xy} \geq 0$ $\forall x \neq y$ iff $p_{xy}(t) \geq 0$ $\forall x, y \in I$, $t \geq 0$.

Assume that Q is a Q-matrix. Then $\sum_y q_{xy} = 0$.

$$\sum_y Q^k_{xy} = \sum_y\sum_z Q^{k-1}_{xz}q_{zy} = \sum_z Q^{k-1}_{xz}\sum_y q_{zy} = 0.$$

Hence $$\sum_y p_{xy}(t) = 1 + \sum_k\frac{t^k}{k!}\underbrace{\sum_y Q^k_{xy}}_{=0} = 1.$$

So $P(t)$ is a stochastic matrix $\forall t \geq 0$.

Now assume $P(t)$ is a stochastic matrix.

$$Q = \left(\frac{d}{dt}\right)\Big|_{t=0} P(t)$$

So, $$\sum_y Q_{xy} = \sum_y\left(\frac{d}{dt}\right)\Big|_{t=0} P(t)_{xy}$$
$$= \left(\frac{d}{dt}\right)\Big|_{t=0}\underbrace{\sum_y P_{xy}(t)}_{\equiv 1}$$
$$= \left(\frac{d}{dt}\right)\Big|_{t=0} 1 = 0.$$

I.e., Q is a Q-matrix. $\square$

__Thm:__ Let X be a right-cont. process with values in a finite set I and let Q be a Q-matrix on I. Then TFAE —

(a) the process X is Markov with generator Q.

(b) (Infinitesimal generator): Conditional on $X_s = x$, the process $(X_{s+t})_{t\geq 0}$ is indep. of $(X_r)_{r\leq s}$ and uniformly as $h \downarrow 0$, $\forall x, y$,
$$\mathbb{P}(X_{t+h} = y \mid X_t = x) = \mathbb{1}(x=y) + q_{xy}h + o(h)$$

(c) $\forall n \geq 0$, $0 \leq t_1 \leq \cdots \leq t_n$ and all states $x_0, \ldots, x_n$:
$$\mathbb{P}(X_{t_n} = x_n \mid X_{t_{1:n-1}} = x_{1:n-1}) = P_{x_{n-1},x_n}(t_n - t_{n-1}),$$

where $(p_{xy}(t))$ is the unique solution to the forward eqn $P'(t) = P(t)Q$ and $P(0) = I$.

__Proof:__ (a)$\iff$(c) follows from previous Thm with countable state space and prev. <u>Prop</u> with finite state space.

To prove (b)$\iff$(c)

__(c)$\Rightarrow$(b):__ $P(t) = e^{tQ}$ is the unique soln. So as $t \downarrow 0$, $P(t) = I + tQ + O(t^2)$.

So, $\mathbb{P}(X_{t+h} = y \mid X_t = x) = p_{xy}(h) = \delta_{xy} + hq_{xy} + \underbrace{O(h^2)}_{o(h)}$

__(b)$\Rightarrow$(c):__ As in P.P.

$$p_{xy}(t+h) = \sum_z p_{xz}(t)\,\mathbb{P}(X_{t+h} = y \mid X_t = z)$$
$$= \sum_z p_{xz}(t)\left(\mathbb{1}(z=y) + q_{zy}h + o(h)\right)$$

So $$\frac{p_{xy}(t+h) - p_{xy}(h)}{h} = \sum_z p_{xz}(t)q_{zy}h + o(1)$$

as $h \downarrow 0$, $p'(t)_{xy} = (P(t)Q)_{xy}\ldots$

<u>Qualitative Properties of Continuous-Time Markov Chains</u>

(Assume chain is minimal and countable state space I.)

<u>Class Structure:</u>

__Def:__ For states $x, y \in I$, write $x \to y$ ("x leads to y") if $\mathbb{P}_x(X_t = y \text{ for some } t \geq 0) > 0$.

We write $x \leftrightarrow y$ ("x communicates with y"), if $x \to y$ and $y \to x$.

Also define <u>communicating classes</u>, <u>irreducibility</u>, <u>closed class</u> and <u>absorbing state</u> exactly as in discrete-time MC.

APPLIED PROBABILITY
LECTURE 9

Prop: Let X be Markov(Q) and transition semigroup $(P(t))_{t\geq 0}$. For any two states $x, y \in I$, TFAE

(a) $x \to y$

(b) $x \to y$ for the jump chain.

(c) $q_{x_0 x_1} \cdots q_{x_{n-1} x_n} > 0$ for some $x = x_0, x_1, \dots, x_n = y$

(d) $p_{xy}(t) > 0$ for all $t > 0$.

(e) $p_{xy}(t) > 0$ for some $t > 0$.

proof: (d) $\Rightarrow$ (e) $\Rightarrow$ (a) $\Rightarrow$ (b) easy.
Remains to prove: (b) $\Rightarrow$ (c) $\Rightarrow$ (d)

(b) $\Rightarrow$ (c): Since $x \to y$ in the jump chain, $\exists$ states $x = x_0, \dots, x_n = y \in I$, (wlog $x_i \neq x_{i+1}$ $\forall i$) s.t. $\tilde{p}_{x_0 x_1} \times \dots \times \tilde{p}_{x_{n-1} x_n} > 0$. But then

$$\tilde{p}_{xy} = \frac{q_{xy}}{q_x} \quad \forall x, y \in I, \text{ as long as } q_x > 0.$$
$$= \mathbb{1}(x=y), \quad q_x = 0 \qquad \text{so} \quad q_{x_i} > 0 \ \forall i$$
$$= 0 \quad (x \neq y)$$

So $q_{x_0 x_1} \cdots q_{x_{n-1} x_n} > 0$

(c) $\Rightarrow$ (d): For any two states w, z with $q_{wz} > 0$, then for any $t > 0$,

$$p_{wz}(t) \geq \mathbb{P}_w(J_1 < t, Y_1 = z, J_2 > t)$$
$$\geq \mathbb{P}_w(J_1 < t, Y_1 = z, S_2 > t)$$
$$= (1 - e^{-q_w t}) \frac{q_{wz}}{q_w} e^{-q_z t}$$

Hence if (c) holds,

$$p_{xy}(t) \geq p_{x_0 x_1}\left(\tfrac{t}{n}\right) p_{x_1 x_2}(t/n) \times \dots \times p_{x_{n-1} x_n}(t/n) > 0.$$

($x_n = y$)

Hitting times: Let $A \subseteq I$, set $T_A = \inf\{t \geq 0 : X_t \in A\}$. ($t > 0$)
Also for the jump chain Y, define $H_A = \inf\{n \geq 0 : X_t \in A\}$. Define
$h_A(x) = \mathbb{P}_x(T_A < \infty)$: hitting prob.
and $k_A(x) = \mathbb{E}_x T_A$: mean hitting prob.
$\{T_A < \infty\} = \{H_A < \infty\}$ and $T_A = J_{H_A}$ on $\{H_A < \infty\}$
(using X is minimal)

Thm: $(h_A(x))_{x\in I}$ and $(k_A(x))_{x \in I}$ are the minimal non-negative solutions to

(h) $\begin{cases} h_A(x) = 1 & \forall x \in A \\ Q h_A(x) = \sum_y q_{xy} h_A(y) = 0 & \forall x \notin A \end{cases}$

(k) $\begin{cases} k_A(x) = 0 & \forall x \in A \\ Q k_A(x) = \sum q_{xy} k_A(y) = -1 & \forall x \notin A. \end{cases}$

Assume $q_w > 0$

Proof: proof of minimality is as in the discrete case, so we only prove that (h_A) & (k_A) solve (h) & (k) respectively.

(h): The hitting prob case the same as in discrete case as for the jump chain.
$h_A(x) = 1 \ \forall x \in A$.
And $\forall x \notin A$, $h_A(x) = \sum_{y \neq x} \tilde{p}_{xy} h_A(y)$

$$= \sum_{y\neq x} \frac{q_{xy}}{q_x} h_A(y)$$

i.e. $q(x) h_A(x) = \sum_{y \neq x} q_{xy} h_A(y)$
$q_x = -q_{xx}$
$$\Rightarrow \sum_y q_{xy} h_A(y) = 0.$$

(k): Clearly if $x \in A$, $k_A(x) = 0$, as $T_A = 0$. Let $x \notin A$. Then $T_A \geq J_1$. Then

$$k_A(x) = \mathbb{E}_x T_A = \mathbb{E}_x J_1 + \mathbb{E}_x(T_A - J_1)$$

So, $k_A(x) = \mathbb{E}_x J_1 + \sum_{z \neq x} \underbrace{\mathbb{E}_x[(T_A - J_1) \mid X_{J_1} = z]}_{= \mathbb{E}_z[T_A] = k_A(z)} \times \tilde{p}_{xz}$, $\tilde{p}_{xz} = \frac{q_{xz}}{q_x}$

So $k_A(x) = \frac{1}{q_x} + \sum_{z \neq x} \frac{k_A(z) q_{xz}}{q_x}$

$$\Rightarrow \sum_z q_{xz} k_A(z) = 1$$

Recurrence and Transience:

$\{n : Y_n = x\}$ is unbounded w.p.1 $\Leftrightarrow$ recurrent
$\{n : Y_n = x\}$ is bdd w.p.1 $\Leftrightarrow$ transient

Def: The state x is called recurrent for x if $\mathbb{P}_x(\{t : X_t = x\}$ is unbounded$) = 1$ and transient if $\mathbb{P}_x(\{t : X_t = x\}$ is unbounded$) = 0$.

Remark: If X explodes with positive probability starting from x, i.e., $\mathbb{P}_x(\zeta < \infty) > 0$, then x cannot be recurrent (as X is minimal)

Thm: Let $X \sim$ Markov(Q) with jump chain Y. Then

(a) If x is recurrent for Y, then x is recurrent for X.

(b) If x is transient for Y, then x is transient for X.

(c) Every state is either recurrent or transient for X.

(d) Recurrence & Transience are class properties.

Proof: (a), (b) & results from discrete time $\Longrightarrow$ (c), (d).

(a) Suppose x is recurrent for Y and $X_0 = x$ then X cannot be explosive, so $\mathbb{P}_x(\zeta = \infty) = 1$ i.e. $J_n \to \infty$ w.p.1. Since $X_{J_n} = Y_n$ $\forall n$ and Y visits x infinitely many times w.p.1, the set $\{t : X_t = x\}$ is unbounded w.p.1.

(b) If x is transient for Y, then $q_x > 0$ (else x is absorbing for Y). As x is transient for Y a.s. $\exists$ a last visit to x for Y, i.e. $N = \sup\{n : Y_n = x\} < \infty$ w.p.1.
Also $S_{N+1} \sim \text{Exp}(q_x) < \infty$ w.p.1.
Thus, $J_{N+1} < \infty$ a.s. and
$\sup\{t : X_t = x\} \leq J_{N+1} < \infty$ a.s.

Like in discrete-time case, x is recurrent iff $\sum_n \tilde{p}_{xx}(n) = \infty$, we have

Thm: x is recurrent for $X \Leftrightarrow \int_0^\infty p_{xx}(t)\,dt = \infty$
x is transient for $X \Leftrightarrow \int_0^\infty p_{xx}(t)\,dt < \infty$

Proof: If $q_x = 0$, nothing to prove.
Assume $q_x > 0$, then

$$\int_0^\infty p_{xx}(t)\,dt = \int_0^\infty \mathbb{P}_x(X_t = x)\,dt = \int_0^\infty \mathbb{E}[\mathbb{1}(X_t = x)]\,dt$$
$$= \mathbb{E}_x\left[\int_0^\infty \mathbb{1}(X_t = x)\,dt\right]$$
$$= \mathbb{E}_x\left[\sum_n \mathbb{1}(Y_n = x) \times S_{n+1}\right]$$
$$= \sum_{n\geq 0} \mathbb{E}_x[\mathbb{1}(Y_n = x) \times S_{n+1}]$$
$$= \sum_{n \geq 0} \mathbb{E}_x[S_{n+1} \mid Y_n = x]\,\mathbb{P}(Y_n = x)$$
$$= \frac{1}{q_x} \sum_{n \geq 0} \tilde{p}_{xx}(n)$$

So result follows from discrete case.

APPLIED PROBABILITY
LECTURE 10

If $q_x > 0$, $\int_0^\infty p_{xx}(t)\,dt = \frac{1}{q_x} \cdot \sum_{n=0}^{\infty} \tilde{p}_{xx}(n)$

Invariant Distributions

For discrete-time MC. Π is an invariant measure iff $\Pi P = \Pi$, where P is the transition matrix and $\Pi_x \geq 0$ $\forall x$. If in addition, $\sum_x \Pi_x = 1$, then π is an invariant distribution. If $Y_0 \sim \pi$, then, $Y_1 \sim \pi, \ldots, Y_k \sim \pi$ $\forall k$.

Recall: let Y be an irreducible discrete-time MC.

Recurrent $\Rightarrow$ invariant-measure ✓

Recurrent $\Rightarrow$ unique invariant measure up to scalar multiplication ✓

Recurrent $\not\Rightarrow$ invariant distribution

(i.e. $\mathbb{E}_x[T_x] < \infty$) Positive-recurrent $\Leftrightarrow$ invariant distribution ✓

Transient $\not\Rightarrow$ no invariant measure (SSRW on $\mathbb{Z}^3$, $\pi_x = 1$ $\forall x$)

In fact, recall

Thm: If Y is a discrete-time MC, irreducible and recurrent $\forall x \in I$. Then
$$\nu^x(y) = \mathbb{E}_x\left[\sum_{n=0}^{H_x - 1} \mathbb{1}(Y_n = y)\right] \text{ where}$$
$$H_x = \inf\{n \geq 1 : Y_n = x\}$$

$(\nu^x(\cdot))$ is invariant $\forall x$ and $0 < \nu^x < \infty$ and $\nu^x(x) = 1$.

Thm 2: let Y be irreducible if λ is any invariant measure with $\lambda(x) = 1$, then $\lambda(y) \geq \nu^x(y)$ $\forall y$. If in addition, Y is recurrent, then $\lambda(y) = \nu^x(y)$ $\forall y$

Def: $X \sim$ Markov(Q) Cont. time MC and let λ be a measure. It is called ~~invariant~~ infinitesimally invariant if $\boxed{\lambda Q = 0}$

Lemma: let X be Markov(Q), and Y be its jump chain. Then Π is invariant for X, iff μ defined by $\mu_x = q_x \pi_x$ is invariant for Y.

Proof: Since $q_x(\tilde{p}_{xy} - \delta_{xy}) = q_{xy}$
$$(\Pi Q)_y = \sum_x \pi_x q_{xy} = \sum_x \pi_x q_x (\tilde{p}_{xy} - \delta_{xy})$$
$$= \sum_x \mu_x (\tilde{p}_{xy} - \delta_{xy})$$
$$= (\mu \tilde{P})_y - \mu_y$$
i.e. $\pi Q = 0$ iff $\mu \tilde{P} = \mu$ is invariant for Y. $\square$

Thm: let X be irreducible and recurrent, with generator Q. Then X has an invariant measure which is unique up to scalar multiplication.

Proof: Assume $|I| > 1$. Since X is irreducible, $q_x > 0$ $\forall x$. Fix $x \in I$. Then the jump chain Y is also irreducible & recurrent. Then $(\nu^x(\cdot))$ is invariant for Y and is unique up to scalar multiples.
Since $q_y > 0$, by the previous lemma, $\pi(y) = \frac{\nu^x(y)}{q_y}$ is invariant for X and is unique up to scalar multiples. $\square$

Thm 2: $X \sim$ Markov(Q) and $|I|$ is finite. Then $\Pi P(t) = P$ $\forall t \geq 0$ iff $\Pi Q = 0$.

Proof: Assume $\Pi Q = 0$. Then
$$\Pi P(t) = \Pi e^{tQ} = \Pi \sum_{k=0}^{\infty} \frac{(tQ)^k}{k!} = \sum_{k=0}^{\infty} \frac{(t\Pi Q)^k}{k!}$$
$$= \Pi I = \Pi.$$
Assume $\Pi P(t) = \Pi$ $\forall t \geq 0$. $Q = \frac{d}{dt}\Big|_{t=0} P(t)$

$$\text{so } \Pi Q = \frac{d}{dt}\Big|_{t=0} \Pi P(t) = \frac{d}{dt}\Big|_{t=0} \Pi = 0.$$

Def: let $T_x = \inf\{t \geq J_1 : X_t = x\}$ be the first return time to x.

Def: A recurrent state x is called positive recurrent if $m_x := \mathbb{E}_x T_x < \infty$. Otherwise, we call it null-recurrent.

Thm: let $X \sim$ Markov(Q) be irreducible. Then TFAE

(a) Every state is positive recurrent.
(b) Some state is positive recurrent.
(c) X is non-explosive and has an invariant distribution λ.

Also, when (c) holds, $\lambda(x) = \frac{1}{q_x m_x}$ $\forall x$.

Lemma: Assume $q_x > 0$, define $\mu^x(y) = \mathbb{E}_x\left[\int_0^{T_x} \mathbb{1}(X_s = y)\,ds\right]$ $\forall y$. Then
$$\mu^x(y) = \frac{\nu^x(y)}{q_y}.$$

Proof:
$$\mu^x(y) = \mathbb{E}_x\left[\int_0^{T_x} \mathbb{1}(X_s = y)\,ds\right]$$
$$= \mathbb{E}_x\left[\sum_{n=0}^{H_x - 1} S_{n+1} \cdot \mathbb{1}(Y_n = y)\right]$$
$$= \mathbb{E}\left[\sum_{n=0}^{\infty} S_{n+1} \cdot \mathbb{1}(Y_n = y, n \leq H_x - 1)\right]$$
$$= \sum_{n=0}^{\infty} \mathbb{E}\left[S_{n+1} \cdot \mathbb{1}(Y_n = y, n \leq H_x - 1)\right].$$
$$\mu^x(y) \overset{*}{=} \sum_{n=0}^{\infty} \mathbb{E}_x\left[S_{n+1} \mid Y_n = y, n \leq H_x - 1\right] \times \mathbb{P}_x(Y_n = y, n \leq H_x - 1).$$

$\{n \leq H_x - 1\}^c = \{H_x \leq n\}$ $\hookrightarrow$ only depends on $Y_0, \ldots, Y_n$.

$$\overset{*}{=} \sum_{n=0}^{\infty} \mathbb{E}_x\left[S_{n+1} \mid Y_n = y\right] \cdot \mathbb{P}_x(Y_n = y, n \leq H_x - 1)$$
$$= \frac{1}{q_y} \cdot \sum_{n=0}^{\infty} \mathbb{E}_x\left[\mathbb{1}(Y_n = y, n \leq H_x - 1)\right]$$
$$= \frac{1}{q_y} \mathbb{E}_x\left[\sum_{n=0}^{H_x - 1} \mathbb{1}(Y_n = y)\right]$$
$$= \frac{1}{q_y} \nu^x(y).$$

APPLIED PROBABILITY
LECTURE 11

Recall:
- $X \sim \text{Markov}(Q, \lambda)$, λ is invariant if $\lambda Q = 0$
- If Y is an irreducible discrete-time MC, and λ is any measure with $\lambda(x) = 1$. Then $\lambda(y) \geq \nu^x(y)$ $\forall y$, where
$$\nu^x(y) = \mathbb{E}_x\Big[\sum_{n=0}^{H_x-1} \mathbb{1}(Y_n = y)\Big]$$
If Y is recurrent, then $\lambda = \nu^x$.
- $X \sim \text{Markov}(Q)$. A recurrent state is called positive recurrent if $\mu_x = \mathbb{E}_x T < \infty$.

- **Lemma:** Assume $q_y > 0$, define
$$\mu^x(y) = \mathbb{E}_x\Big[\int_0^{T_x} \mathbb{1}(X_s = y)\,ds\Big] \quad \forall y.$$
Then $\mu^x(y) = \dfrac{\nu^x(y)}{q(y)}$.

Proof (Thm from last time):
(a) $\Rightarrow$ (b) obvious.
(b) $\Rightarrow$ (c): Assume $q_x > 0$ $\forall x$ by irreducibility. Let x be a positive recurrent state. Then by irreducibility, all states are recurrent. So the jump chain Y is recurrent. Hence X is non-explosive, i.e. $\mathbb{P}(\zeta = \infty) = 1$, $\forall y \in I$.

As Y is recurrent, ν^x is an invariant measure for Y. Hence,
$\mu^x(y) := \dfrac{\nu^x(y)}{q(y)}$ is an invariant measure for X.
So enough to show $\sum_{y \in I} \mu^x(y) < \infty$.
$$\sum_{y\in I} \mu^x(y) = \sum_{y \in I} \mathbb{E}_x\Big[\int_0^{T_x} \mathbb{1}(X_s = y)\,ds\Big]$$
$$= \mathbb{E}_x\Big[\int_0^{T_x} \sum_{y\in I} \mathbb{1}(X_s = y)\,ds\Big]$$
But $\sum_{y \in I} \mathbb{1}(X_s = y) = 1$ on $s < \zeta = \infty$.
Thus, $\sum_{y\in I} \mu^x(y) = \mathbb{E}_x[T_x] = m_x < \infty$.

(c) $\Rightarrow$ (a): By the above lemma, the measure $\beta(y) = \lambda(y) q_y$ is invariant for Y.
Also, $\sum_{y \in I} \lambda(y) = 1$, hence $\lambda(x) > 0$ for some x. To show first that $\lambda(y) > 0$ $\forall y$. Fix any $y \in I$. Since X and hence Y is irred., $\tilde{p}_{xy}(n) > 0$ for some $n \in \mathbb{N}$. As $\beta \tilde{P} = \beta$, so $\beta \tilde{P}^n = \beta$, hence
$$\beta_y = \lambda(y) q_y = \sum_z \beta_z \tilde{p}_{zy}(n) \geq \beta_x \tilde{p}_{xy}(n) = \lambda(x) q_x \tilde{p}_{xy}(n) > 0$$
Thus, $\beta_y = \lambda(y) q_y > 0$, i.e. $\lambda(y) > 0$.

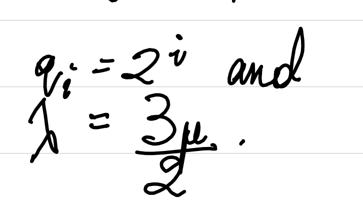
Now, fix any state $x \in I$. Then to show that $\mathbb{E}_x T_x = m_x < \infty$.

Then $\lambda(x) > 0$. Define $a^x(y) = \dfrac{\beta(y)}{\lambda(x) q_x}$, $y \in I$.

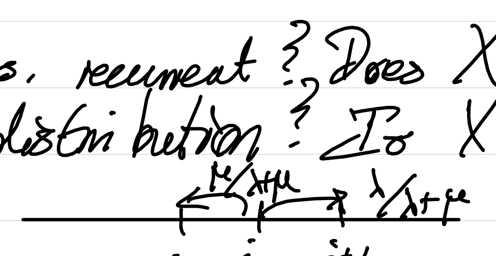
Then $(a^x(\cdot))$ is invariant and $a^x(x) = 1$. Then by the thm for discrete-time chain, $a^x(y) \geq \nu^x(y)$ $\forall y$. As before by lemma (*), $\mu^x(y) = \dfrac{\nu^x(y)}{q(y)}$ & $m_x = \sum_{y \in I} \mu^x(y)$
(as x is non-explosive).
So, $m_x = \sum_y \mu^x(y) = \sum_y \frac{1}{q_y} \nu^x(y)$
$$\leq \sum_y \frac{a^x(y)}{q_y} = \frac{1}{q_x \lambda(x)} \sum_y \lambda(y)$$
$$= \frac{1}{q_x \lambda(x)} < \infty.$$

Also, when (c) holds, all states are positive-recurrent, hence recurrent, so Y is recurrent. Hence $a^x(y) = \nu^x(y)$ $\forall y$.
Hence, $m_x = \dfrac{1}{\lambda(x) q_x}$, i.e. $\lambda(x) = \dfrac{1}{m_x q_x}$.

□

Example: X on $\mathbb{Z}^+$ with
$$\begin{cases} q_{i,i+1} = \lambda q_i \\ q_{i,i-1} = \mu q_i \\ q_{i,j} = 0 \text{ otherwise} \end{cases}$$


with $q_i = 2^i$ and $\lambda = \dfrac{3\mu}{2}$.

Is X pos. recurrent? Does X have an invariant distribution? Is X explosive?
Y on $\mathbb{Z}^+$:


is a birth-death chain. So $(\lambda/\mu)^i$ is an invariant measure for Y. So
$\pi_i = \dfrac{1}{q_i}\Big(\dfrac{\lambda}{\mu}\Big)^i$ is an invariant measure for X.
So $\pi_i = (3/4)^i$
Since $\sum_i (3/4)^i < \infty$, X has an invariant distribution.
So Y is transient (as $\lambda > \mu$).
So X is transient. Hence, not positive recurrent. So by the last thm, X is explosive.

Lemma: Set $Z_n = X_{nt}$ where X is a cont. time MC and $t > 0$ fixed. Then x is recurrent for X iff it is recurrent for $(Z_n)_{n \in \mathbb{N}}$.

Proof: Ex. Sheet 2.

Thm: Let $X \sim \text{Markov}(Q)$ be irreducible, recurrent and λ be a measure. Then
$$\lambda Q = 0 \iff \lambda P(s) = \lambda \quad \forall s > 0.$$

Proof: As X is recurrent, so is Y, hence $(\nu^x(\cdot))$ is invariant for Y for any fixed $x \in I$. Hence
$\mu^x(y) = \dfrac{\nu^x(y)}{q_y}$, $(\mu^x(\cdot))$ is invariant for X, i.e. $\mu^x Q = 0$.

We will next show that $\mu^x P(s) = \mu^x$ $\forall s > 0$. Fix any $s > 0$. By the strong Markov Property,
$$\mathbb{E}_x\Big[\int_0^s \mathbb{1}(X_t = y)\,dt\Big]$$
$$= \mathbb{E}_x\Big[\int_{T_x}^{T_x+s} \mathbb{1}(X_t = y)\,dt\Big]$$
(as $T_x < \infty = \zeta$ a.s.)
Thus, $\forall y \in I$,
$$\mu^x(y) = \mathbb{E}_x\Big[\int_0^{T_x} \mathbb{1}(X_t = y)\,dt\Big]$$
$$= \mathbb{E}_x\Big[\int_0^s \mathbb{1}(X_t = y)\,dt\Big] + \mathbb{E}_x\Big[\int_s^{T_x} \mathbb{1}(X_t = y)\,dt\Big]$$
$$= \mathbb{E}_x\Big[\int_{T_x}^{T_x+s} \mathbb{1}(X_t = y)\,dt\Big] + \mathbb{E}_x\Big[\int_s^{T_x} \mathbb{1}(X_t = y)\,dt\Big]$$
$$= \mathbb{E}_x\Big[\int_0^{T_x+s} \mathbb{1}(X_t = y)\,dt\Big]$$
$$= \mathbb{E}_x\Big[\int_0^\infty \mathbb{1}(X_{u+s} = y, u < T_x)\,du\Big]$$
$$= \int_0^\infty \mathbb{P}(X_{u+s} = y, u < T_x)\,du.$$
$$= \int_0^\infty \sum_{z \in I} \mathbb{P}(X_u = z, X_{u+s} = y, u < T_x)\,du.$$
$$= \int_0^\infty \sum_{z\in I} \mathbb{P}(X_{u+s} = y \mid \underbrace{u < T_x}_{\text{depends only on } (X_s)_{s \leq u}}, X_u = z)\,\mathbb{P}(u < T_x, X_u = z)\,du$$
$$\overset{\text{Markov}}{=} \int_0^\infty \sum_{z \in I} \mathbb{P}(X_{u+s} = y \mid X_u = z) \times \mathbb{P}(u < T_x, X_u = z)\,du$$
$$= \int_0^\infty \sum_{z\in I} p_{zy}(s)\,\mathbb{P}(u < T_x, X_u = z)\,du$$
$$= \sum_{z \in I} p_{zy}(s)\,\mathbb{E}_x\Big[\int_0^\infty \mathbb{1}(u < T_x, X_u = z)\,du\Big]$$
$$= \sum_{z\in I} p_{zy}(s)\,\mu^x(z)$$
i.e. $\mu^x = \mu^x P(s)$.

Lemma: Set $Z_n = X_{nt}$ for some fixed $t>0$, where X is a continuous MC. Then x is recurrent for X iff x is recurrent for Z.

Pf: See ES2.

Thm: Let $X \sim \text{Markov}(Q)$ be irreducible and recurrent, and λ be any measure. Then $\lambda Q = 0$ iff $\lambda P(s) = \lambda$ $\forall s>0$

Proof: With $\mu^x(y) = \mathbb{E}_x\left[\int_0^{T_x} \mathbb{1}(X_t = y)\,dt\right]$, we had shown $\mu^x Q = 0$ and
$$\mu^x P(s) = \mu^x \quad \forall s>0.$$

Now, as x is recurrent, any measure λ that satisfies $\lambda Q = 0$, is unique up to scalar multiples. (Thm 1 for invariant meas.)

Also, any measure λ satisfying $\lambda P(s) = \lambda$ is unique up to scalar multiples.
(Fix $s>0$. $P(s)$ is the transition matrix for the discrete-time MC Z, where $Z_n = X_{ns}$. Z is irreducible and recurrent by the previous lemma with λ being an invariant measure, hence unique up to scalar multiples.)

Thus $\lambda Q = 0 \iff \lambda$ is a scalar multiple of μ^x.
$\iff \lambda P(s) = \lambda$ $\forall s>0$. $\square$

Remark: In fact, if X is irreducible and $\pi P(t) = \pi$ $\forall t$ for some distribution π, then $\pi Q = Q$

Pf: Ex.

Convergence to Equilibrium:

For a discrete-time MC Y which is irreducible, positive-recurrent and aperiodic ($\gcd\{n : p_{xx}(n) > 0\} = 1$),
$$\mathbb{P}_x(Y_n = y) \to \pi(y) \text{ as } n\to\infty, \forall x,$$
where π is the inv. distribution of Y.

Thm: Let $X \sim \text{Markov}(Q)$ be irreducible, non-explosive and let λ be an invariant distribution. Then $\forall x,y \in I$
$$p_{xy}(t) \longrightarrow \lambda \text{ as } t\to\infty.$$

Lemma: For the semi-group $P(t)$:
$$|p_{xy}(t+h) - p_{xy}(t)| \le 1 - e^{-q_x h} \le q_x h.$$
$e^{x} \ge 1+x$ $\forall x$

Proof: $|p_{xy}(t+h) - p_{xy}(t)| = \left|\sum_z p_{xz}(h) p_{zy}(t) - p_{xy}(t)\right|$

$$\circledast\ \left|\sum_{z\ne x} p_{xz}(h)\underbrace{p_{zy}(t)}_{\in[0,1]} - p_{xy}(t)(1-p_{xx}(h))\right|$$

$$\sum_{z\ne x} p_{xz}(h) p_{zy}(t) \le \sum_{z \ne x} p_{xz}(h) = 1 - p_{xx}(h)$$

$$p_{xy}(t)(1-p_{xx}(h)) \le 1 - p_{xx}(h).$$

So $\circledast \le 1 - p_{xx}(h) \le \mathbb{P}_x(J_1 \le h) = 1 - e^{-q_x h}$ $\square$

Proof of Thm: Fix $\varepsilon > 0$. Fix $h>0$ s.t. $q_x h \le \frac{\varepsilon}{2}$.
Let $(Z_n)_{n\ge0} = (X_{nh})_{n\ge0}$.

Then $(Z_n)_{n\ge0}$ irreducible and aperiodic (as $p_{xy}(h) > 0$ $\forall x,y \in I$).
As X is positive recurrent, so $\lambda P(h) = \lambda$ (by prev. Thm), so λ is the invariant distribution for Z.
Then, by discrete-time MC result, $\forall x, y$
$$p_{xy}(nh) \xrightarrow[n\to\infty]{} \lambda(y) \quad \forall x,y.$$

Hence $\exists n_0$ s.t. $\forall n \ge n_0$, $|p_{xy}(nh) - \lambda(y)| < \frac{\varepsilon}{2}$
Let $t \ge n_0 h$, $nh \le t \le (n+1)h$ for some $n \ge n_0$.
Thus,
$$|p_{xy}(t) - \lambda(y)| \le |p_{xy}(t) - p_{xy}(nh)| + |p_{xy}(nh) - \lambda(y)|$$
$$\overset{\text{by prev. lemma}}{\le} q_x(t-nh) + \varepsilon/2 \le q_x h + \varepsilon/2 < \varepsilon.$$

Ergodic Theorems:

Thm: Let Q be an irred. Q-matrix, and $X \sim \text{Markov}(\nu, Q)$. Then, almost surely,
$$\frac{1}{t}\int_0^t \mathbb{1}(X_s = x)\,ds \longrightarrow \frac{1}{m_x q_x}, \text{ as } t\to\infty.$$
where $m_x = \mathbb{E}_x T_x$. Also in the pos. recurrent case, with π being the invariant distribution, if $f: I \to \mathbb{R}$ bounded, then a.s.
$$\frac{1}{t}\int_0^t f(X_s)\,ds \xrightarrow[t\to\infty]{} \mathbb{E}_\pi f = \sum_x f(x)\pi(x).$$

Reversibility:

Thm: Let $X \sim \text{Markov}(Q)$ irred., non-explosive with invariant distribution π.
Let $X_0 \sim \pi$. Fix $T>0$ and set $\hat{X}_t = X_{T-t}$, for $0 \le t \le T$.
Then, $\hat{X} \sim \text{Markov}(\hat{Q})$, and invariant distribution π, where
$$\hat{q}_{xy} = \frac{\pi(y)\, q_{yx}}{\pi(x)}$$
Also, $\hat{Q}$ is irred & non-explosive (i.e. if $Z \sim \text{Markov}(\hat{Q})$ is non-explosive).

Proof: Note that $\hat{Q}$ is a Q-matrix as $\hat{q}_{xy} \ge 0$, $x \ne y$ and $\sum_y \hat{q}_{xy} = \frac{1}{\pi_x}\sum_y \pi_y q_{yx}$
$$= \frac{1}{\pi_x}(\pi Q)_x = 0$$
$(\pi\hat{Q})_y = \sum_x \pi_x \hat{q}_{xy} = \pi(y)\sum_x q_{yx} = 0$, i.e. π is invariant for $\hat{Q}$.

Now, let $0 = t_0 \le t_1 \le \dots \le t_n = T$ and $x_0, x_1, \dots, x_n \in I$: then

$$\mathbb{P}(\hat{X}_{t_0} = x_0, \dots, \hat{X}_{t_n} = x_n) = \mathbb{P}(X_T = x_0, \dots, X_{T-t_{n-1}} = x_{n-1}, X_0 = x_n)$$

Let $s_i = t_i - t_{i-1}$, $\circledast = \pi(x_n)\, p_{x_n x_{n-1}}(s_n) \cdots p_{x_1 x_0}(s_1)$

Define $\hat{p}_{xy}(t) = p_{yx}(t)\frac{\pi_y}{\pi_x}$

Then $\circledast\ \pi(x_n)\,\hat{p}_{x_{n-1}x_n}(s_n)\frac{\pi(x_{n-1})}{\pi(x_n)} \cdots \frac{\pi(x_0)}{\pi(x_1)}\hat{p}_{x_0 x_1}(s_1)$
$$= \pi(x_0)\,\hat{p}_{x_0x_1}(s_1) \times \dots \times \hat{p}_{x_{n-1}x_n}(s_n)$$

So $\hat{X}$ is Markov with transition semi-group $(\hat{P}(t))$.
Need to show that $\hat{P}(t)$ is the minimal non-negative solution to Kolmogorov backward equation with $\hat{Q}$.

Indeed, $\hat{p}'_{xy}(t) = \frac{\pi(y)}{\pi(x)} p'_{yx}(t) = \frac{\pi(y)}{\pi(x)}\sum_z p_{yz} q_{zx}$ (since $P'(t) = P(t)Q$).
$$= \frac{1}{\pi(x)}\sum_z \pi(z)\,\hat{p}_{zy}(t)\, q_{zx}$$
$$= \frac{1}{\pi(x)}\sum_z \pi(x)\,\hat{p}_{zy}(t)\,\hat{q}_{xz} = \sum_z \hat{p}_{zy}(t)\,\hat{q}_{xz}$$

Suppose R is another solution to the backward eq. with $\hat{Q}$. Then
$R'(t) = \hat{Q}P(t)$. Then with $\bar{R}_{xy}(t) = R_{yx}(t)\frac{\pi(y)}{\pi(x)}$
then R satisfies $\bar{R}'(t) = \bar{R}(t)Q$ as before,
hence $\bar{R} \ge P$, thus $R \ge \hat{P}$.

Reversibility:

Thm: Let $X \sim \text{Markov}(Q)$ be irreducible, non-explosive with inv. distribution π. Let $X_0 \sim \pi$. Fix $T>0$ and set $\hat{X}_t = (X_{T-t})$ for $0 \le t \le T$. Then $\hat{X} \sim \text{Markov}(\hat{Q})$ and inv. distr. π, where $\hat{q}_{xy} = q_{yx}\frac{\pi(y)}{\pi(x)}$. Also $\hat{Q}$ is irreducible and non-explosive.

Pf: $\hat{Q}$ is irred. ✓ π inv. for $\hat{Q}$ ✓
$\hat{p}_{xy}(t) = p_{yx}(t)\frac{\pi(y)}{\pi(x)}$ and $\hat{X}$ is Markov with semi-group $\hat{p}$ satisfies the Kolmogorov backward equation with $\hat{Q}$ and was minimal.

To show that $\hat{Q}$ is non-explosive.
Since X irred., non-explosive with inv. dist π
$\Rightarrow X$ is pos. recurrent $\Rightarrow$ recurrent
$\Rightarrow \pi P(t) = \pi \;\; \forall t$

Thus, $\sum_y \hat{p}_{xy}(t) = \sum_y \frac{\pi(y)}{\pi(x)} p_{yx}(t)$
$= \frac{1}{\pi(x)} \sum_y \pi(y) p_{yx}(t) = \frac{1}{\pi(x)} \pi(x) = 1.$

So $\mathbb{P}_x(\hat{\zeta} > t) = \sum_y \hat{p}_{xy}(t) = 1$, so taking $t \uparrow \infty \Rightarrow \hat{\zeta} = \infty$ a.s., i.e. $\hat{X}$ is non-explosive. $\square$

Def: Let $X \sim \text{Markov}(Q)$. It is called reversible if $\forall T>0$, $(X_t)_{0\le t\le T}$ and $(X_{T-t})_{0 \le t \le T}$ have the same distribution.

Def: A measure λ and Q-matrix Q are said to be in detailed balance if $\forall x, y \in I$
$\lambda(x) q_{xy} = \lambda(y) q_{yx}$

Lemma: If Q and λ are in detailed balance, then λ is invariant for Q.

Pf: $(\lambda Q)_y = \sum_x \lambda(x) Q_{xy} = \sum_x \lambda(y) Q_{yx} = \lambda(y) \sum_x Q_{yx} = 0.$

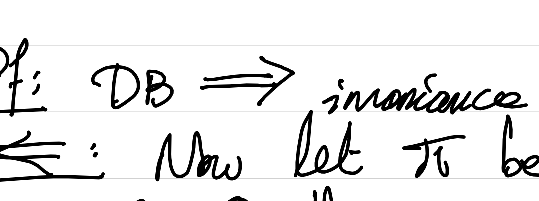
Remark: To find an invariant distribution, check if the detailed balanced eq. are satisfied as a first step.

Lemma: Let $X \sim \text{Markov}(Q)$ be irred., non-explosive and π be a distribution s.t. $X_0 \sim \pi$. Then π and Q are in detailed balance (DB) iff $(X_t)_{t\ge 0}$ is reversible.

Pf: Assume π & Q are in D.B. $\Rightarrow$
π is invariant for Q by prev. lemma and $\hat{Q} = Q$
$\Rightarrow$ by previous thm that (X) is reversible.

Assume (X_t) is reversible $\Rightarrow X_t \overset{d}{=} X_0 \; \forall t$ and $X_0 \sim \pi \Rightarrow \pi P(t) = \pi \;\; \forall t$ and π a distribution
$\Rightarrow \pi Q = 0 \Rightarrow Q = \hat{Q}$ by prev. thm
$\Rightarrow \pi$ & Q are in D.B.

Def: A birth and death chain X is a cont. time Markov chain on $\{0,1,2,\dots\}$ with

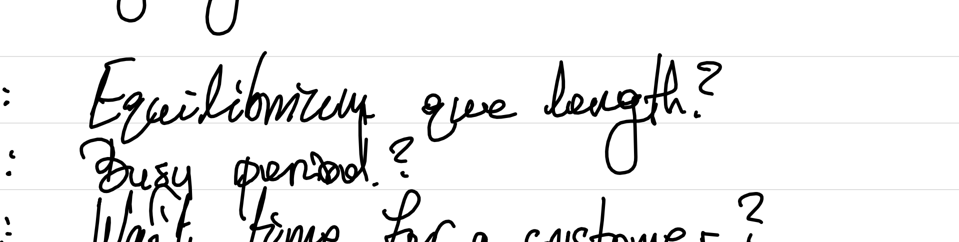


$$\begin{cases} q_{x,x-1} = \mu_x \\ q_{x,x+1} = \lambda_x \\ q_{x,x} = -(\mu_x + \lambda_x) \\ q_{x,y} = 0 \;\; \forall y \ne x-1, x+1 \\ \& \; q_{0,1} = \lambda_0, \; q_{0,0} = -\lambda_0 \end{cases}$$

Lemma: A measure π is invariant for a B&D chain iff it solves the detailed balance.

Pf: DB $\Rightarrow$ invariance follows from prev. lemma.
$\Leftarrow$: Now let π be an invariant measure, i.e. $\pi Q = 0$ then
$\forall j \ge 1 \;\; \sum_i \pi_i q_{ij} = 0 \Rightarrow \pi_{j-1} q_{j-1,j} + \pi_j q_{j,j} + \pi_{j+1} q_{j+1,j} = 0$
$\Rightarrow \pi_{j-1}\lambda_{j-1} + \pi_{j+1}\mu_{j+1} = \pi_j(\lambda_j + \mu_j)$
$\Rightarrow \pi_{j+1}\mu_{j+1} - \pi_j \mu_j = \pi_j \lambda_j - \pi_{j-1}\lambda_{j-1}$
$= \dots = \pi_1 \mu_1 - \pi_0 \lambda_0$ (by induction)
Also for $j=0$, $\sum \pi_i q_{i,0} \Rightarrow \pi_1\mu_1 - \pi_0\lambda_0 = 0.$
So $\pi_{j+1}\mu_{j+1} = \pi_j \lambda_j \;\; \forall j \ge 0.$ $\square$

Queuing Theory:



$X_t :=$ length of queue at time t.
(includes ppl in queue and those getting served).

Q: Equilibrium que length?
Q: Busy period?
Q: Wait time for a customer?
(in queue time + service time, i.e. time spent in the system)

Kendall's notation: $G/G/K$
Arrival process, Service distribution, # servers (usually 1 or ∞)

G: general
M: Markovian / memoryless, i.e. P.P.(λ) or iid Exp.

M/G/1: Arrival process P.P.(λ), Service dist. iid. with dist G } ind.
1 server

M/M/1: Arrival process P.P.(λ), Service dist. iid. Exp(μ) } ind.
1 server

Let X_t be the queue length (includes customers being served).

$I = \{0, 1, 2, \dots\}$

In particular, when both arrival is P.P.(λ) and service is iid Exp(μ), (X_t) is a Markov chain (by the memorylessness ppty).
Also, M/M/1 : } are B&D chains
M/M/∞ : }

M/M/1: $q_{i,i+1} = \lambda$, $q_{i,i-1} = \mu$.
M/M/∞: $q_{i,i+1} = \lambda$, $q_{i,i-1} = i\mu$.
customers served.

Thm: (M/M/1): Let $\rho = \lambda/\mu$. Then the queue length X is transient $\iff \rho > 1$
recurrent $\iff \rho \le 1$
pos. rec. $\iff \rho < 1$.

In the pos. rec. case, the inv. dist. is $\pi(n) = (1-\rho)\rho^n$, $n = 0, 1, 2, \dots$
And if $\rho < 1$, $X_0 \sim \pi$, then the wait time for a customer that arrives at time t is Exp$(\mu - \lambda)$.

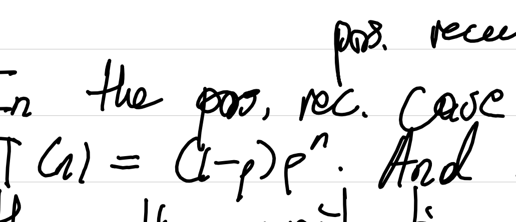
Recall: $X = \sum_{i=1}^{T} X_i$ where X_i iid Exp(μ) & $T \sim \text{Geo}(p)$ ind. of (X_i). Find the dist. of X.

APPLIED PROBABILITY
LECTURE 14

Queues:

Thm (M/M/1): Let $\rho = \lambda/\mu$. Then the queue length X is: transient $\iff \rho > 1$
recurrent $\iff \rho \le 1$
pos. recurrent $\iff \rho < 1$.

In the pos. rec. case, the inv. dist. is $\pi(n) = (1-\rho)\rho^n$. And if $\rho < 1$, and $X_0 \sim \pi$, then the wait time for a customer arriving at time t is $Exp(\mu - \lambda)$.



1 server (iid $Exp(\mu)$).

$\begin{cases} q_{i,i-1} = \mu \\ q_{i,i+1} = \lambda \end{cases}$, $q_{i,j} = 0$ otherwise.

The jump chain chain Y is a biased SRW on $\{0, 1, 2, \dots\}$

Thus, Y (and hence X) is transient iff $\lambda > \mu$
" " recurrent iff $\lambda \le \mu$.

This X is non-explosive as $\sup_i q_i = \lambda + \mu < \infty$
So pos. rec. $\iff \exists$ an invariant distribution.

As X is a birth-death chain, $\exists$ an invariant measure $\iff$ detailed balance eqn is satisfied.

$\pi(n+1)\mu = \pi(n)\lambda, \ \forall n \ge 0$ (DB)

So, $\pi(n+1) = (\lambda/\mu)\pi(n) = \cdots (\lambda/\mu)^{n+1}\pi(0)$.

Thus, π is summable iff $\lambda/\mu < 1$. Then, when $\lambda/\mu < 1$, $\pi(n) = (1-\rho)\rho^n$, $n = 0, 1, 2, \dots$

So π is the (shifted) geometric r.v.; i.e. π is the dist. of $Z - 1$, where $Z \sim Geom(1-\rho)$.

If $\rho < 1$, $X_0 \sim \pi$, then $X_t \sim \pi$ as X is recurrent & π is inv. So the wait time for the customers is

$$\sum_{i=1}^{X_t+1} T_i, \text{ where } T_i \text{ iid } Exp(\mu) \text{ indep. of } X_t.$$

$X_t + 1 = Z \sim Geo(1-\rho)$. So

$\sum_{i=1}^{Z} T_i \sim Exp(\mu(1-\rho))$ by ES1. □

Remark: $\mathbb{E}_\pi X_t = \mathbb{E}(Z-1) = \frac{1}{1-\rho} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$

arrival rate

Expected queue length at equilibrium.

Avg. wait of a customer in equilibrium.

"Long-run avg. queuelength
= Long-run avg. wait time
× Arrival rate."

⚠ This is called Little's formula
(Very general formula)

Thm (M/M/∞): The queue length X_t is pos. recurrent $\forall \mu, \lambda > 0$ with the inv. distribution $Po(\rho)$ where $\rho = \lambda/\mu$.

rates: $\begin{cases} q_{i,i+1} = \lambda \\ q_{i,i-1} = i\mu \\ q_{i,j} = 0, \text{ otherwise} \end{cases}$

Proof: This is a birth-death chain, so we solve the (DB) eqns

$\pi(n) \cdot n\mu = \pi(n+1) \cdot \lambda$ —— (DB).

So, $\pi(n) = \frac{1}{n} \cdot \left(\frac{\lambda}{\mu}\right) = \cdots = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \pi(0)$

So, π is always summable, so after normalising, $\pi(n) = e^{-\lambda/\mu}\frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}$

i.e. $\pi(n) \sim Po(\rho)$, where $\rho = \lambda/\mu$.

X is pos. rec. if we show that X is non-explosive. We show that the jump chain Y is recurrent.

Easiest way is to show Y is pos. recurrent, i.e. Y has an invariant distribution.

But μ is an inv. measure for Y where $\mu_n = q_n \cdot \pi_n = e^{-\rho}\frac{\rho^n}{n!}(n\mu + \lambda)$

i.e., $\mu_n = e^{-\rho}\frac{\rho^n}{n!}(n\mu + \mu\rho) = \frac{e^{-\rho}\rho^n}{n!}\mu(n+\rho)$

$= \mu \cdot e^{-\rho}\frac{\rho^n}{(n-1)!} + \mu e^{-\rho}\frac{\rho^{n+1}}{n!}$

So, $\sum_n \mu_n < \infty$, hence Y is has an inv. distribution.

Pos. rec. for $Y \implies$ rec. for $Y \implies X$ non-explosive

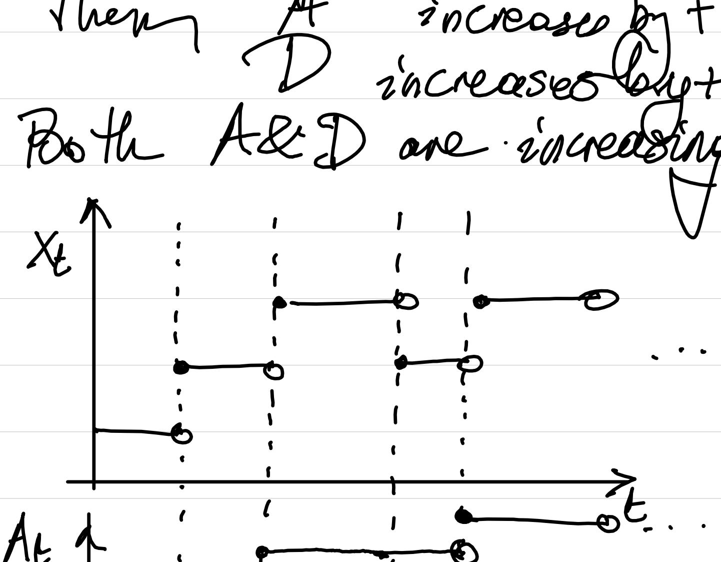
□

⚠ Burke's Theorem: States that the output of a stationary M/M/k queue is again a Poisson process (λ).
(1956, "Output of Queueing systems", conjectured by O'Brien (1954) & Morse (1955)).

Rk: Let A and D denote the arrival and departure processes associated with a queue X, i.e. A_t, D_t: # customers who arrived/departed until time t.

Then A increases by +1, when X does
D increases by +1, when X decreases.

Both A & D are increasing processes (starting from 0).



$X_t = X_0 + A_t - D_t$

Rk: A Poisson process does not have an invariant distribution, but still has the following time-reversal property:

If $N \sim P.P.(\lambda)$, then for any $T \ge 0$, $\hat{N}_t = N(T) - N(T-t)$ is again a Poisson process (λ) on $[0, T]$.

Indeed, conditioning on $N(T) = n$, the distribution of ordered jump times is $\frac{n!}{T^n}\mathbb{1}(0 < t_1 < \dots < t_n < T)$.

Thm (Burke): Consider an M/M/1 queue with $\mu > \lambda > 0$ or an M/M/∞ queue with $\mu, \lambda > 0$. Then at equilibrium, (i.e. when $X_0 \sim \pi$), D is a Poisson process of rate λ and X_t is indep. of $(D_s : 0 \le s \le t)$.

Rk: $X_0 \sim \pi$ is essential. Suppose that $X_0 = 2$, then the first departure happens at $Exp(\mu)$ & not $Exp(\lambda)$ time.

Rk: Clearly the processes $(X_s : s \le t)$ & $(D_s : s \le t)$ are not indep.

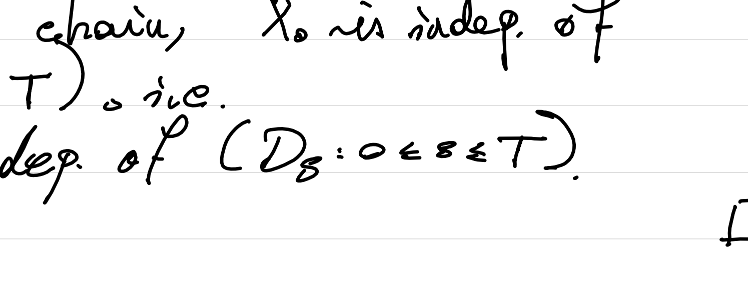
Proof (Burke's thm): As X is a B-D chain, π satisfies the DB eqns. Also, X is irred., non-explosive, so if $X_0 \sim \pi$, then X is reversible. Thus, for a fixed $T>0$, set $\hat{X}_t = X_{T-t}$, $t \in [0,T]$, we have $(\hat{X}_t : 0 \le s \le T) \overset{(d)}{=} (X_t : 0 \le s \le T)$.

$\Longrightarrow$ the arrival process $\hat{A}$ of $\hat{X}$ (until time t) is P.P.(λ).
But, $\hat{A}_t = D_T - D_{T-t} \sim$ P.P.(λ).
Since the time-reversal of P.P.(λ) on $[0,T]$ is again a P.P.(λ), we get $(D_t : 0 \le s \le T)$ is P.P.(λ).
Since $T>0$ is arbitrary, this determines the f.d.d. of $(D_t)_{t \ge 0}$ and so determines the process D. So D is P.P.(λ).

Also, X_0 is indep. of $(A_s : 0 \le s \le T)$. So for the reversed chain, $\hat{X}_0$ is indep. of $(\hat{A}_s : 0 \le s \le T)$, i.e. X_t is indep. of $(D_s : 0 \le s \le T)$.

$\square$

Queues in tandem:



Suppose there is an M/M/1 queue with parameters λ and μ. After a customer is served, they immediately join a second single server queue with some rate μ_2 (all servers distributions are ind.).

Let X and Y denote the queue lengths of the 2 queues respectively. Consider (X,Y). So $I = (\mathbb{N} \cup \{0\})^2$ and the rates

$$(m,n) \longrightarrow \begin{cases} (m+1,n) & \text{w. rate } \lambda \\ (m-1,n+1) & \text{w. rate } \mu_1 \quad \text{if } m,n \ge 1. \\ (m,n-1) & \text{w. rate } \mu_2. \end{cases}$$

Thm: (X,Y) is pos. recurrent iff $\lambda < \mu_1$ and $\lambda < \mu_2$, and then the inv. distribution is given by
$$\pi(m,n) = (1-\rho_1)\rho_1^m (1-\rho_2)\rho_2^n, \quad \rho_1 = \lambda/\mu_1, \ \rho_2 = \lambda/\mu_2$$
thus, at equilibrium, X_t & Y_t are indep.

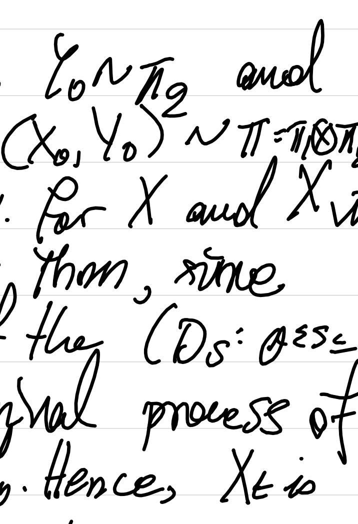
Proof 1: Directly check $\pi Q = 0$. As the rates are bdd, (X,Y) is non-explosive & hence pos. recurrent.

Proof 2: Note the marginal X is an M/M/1 queue thus X is pos. rec iff $\lambda < \mu_1$ with inv. dist. $\pi^1(m) = (1-\rho_1)\rho_1^m$. By Burke's thm, if $X_0 \sim \pi^1$, then the departure process of X, which is the arrival process of $Y \sim$ P.P.(λ), so the marginal Y is again an M/M/1 queue with invariant distribution $\pi^2(n) = (1-\rho_2)\rho_2^n$.

Independence: if $X_0 \sim \pi_1$, $Y_0 \sim \pi_2$ and X_0, Y_0 are indep., then $(X_0, Y_0) \sim \pi = \pi_1 \otimes \pi_2$ and $X_t \sim \pi^1$ (as π^1 is inv. for X and X is recurrent). And by Burke's thm, since $X_0 \sim \pi^1$, X_t is indep. of the $(D_s : 0 \le s \le t)$ of X, which is the arrival process of Y. Also, X_t is ind. of Y_0. Hence, X_t is indep. of Y_t. Also, $Y_t \sim \pi_2$, i.e. $(X_t, Y_t) \sim \pi^1 \otimes \pi^2 = \pi$. This shows that π is inv. for (X,Y) (as $\pi P_t = \pi$ $\forall t \ge 0$ and π is a <u>distribution</u> $\Rightarrow \pi$ is invariant).

Jackson's Networks: A network of N single-server queues, with arrival rates λ_k and service rates μ_k, for $k=1,2,\ldots,N$. After service, each customer in queue i moves to queue j with prob p_{ij} or exits the system with prob $p_{i0} = 1 - \sum_j^N p_{ij}$.

We assume $p_{i0} > 0$ $\forall i=1,2,\ldots,N$ and $p_{ii}=0$ $\forall i$ and the system is irreducible in the sense that a customer arriving in queue i has a pos. prob. to visit queue j at some later time, $\forall i=1,2,\ldots,N$.



Consider $(X_1, \ldots, X_N)$ where X_i denotes the # customers in queue i.
thus $I = (\mathbb{N} \cup \{0\})^N$.
If $n = (n_1, \ldots, n_N)$ and $e_i = (0, \ldots, 1, 0, \ldots, 0)$ ($\leftarrow$ ith position), then

$$\begin{cases} q_{n,n+e_i} = \lambda_i, & i=1,2,\ldots,N \\ q_{n,n-e_i+e_j} = \mu_i p_{ij}, & i,j=1,2,\ldots,N,\ i \ne j \text{ and } n_i \ge 1 \\ q_{n,n-e_i} = \mu_i p_{i0}, & i=1,2,\ldots,N \text{ and } n_i \ge 1. \end{cases}$$

Def: We say a vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_N)$ satisfies the traffic equations if $\forall 1 \le i \le n$, $\bar{\lambda}_i = \lambda_i + \sum_{j=1, j \ne i}^{N} \bar{\lambda}_j p_{ji}$. (*)

Rk: $\bar{\lambda}_i$ is the "effective arrival rate" at queue i at equilibrium.

Lemma: There exists a unique soln to (*)
proof: (Existence): let $p_{00} = 1$. Then $(p_{ij})_{i,j=1}^N$ is a stochastic matrix. The corresponding discrete-time MC (Z_n) is absorbing at 0. Thus the communicating class $\{1,2,\ldots,N\}$ is transient as it is not closed. Thus, if V_i = # visits to state i, then $\mathbb{E}[V_i] < \infty$ $\forall i=1,2,\ldots,N$ starting from any initial distribution. Let $\mathbb{P}(Z_0 = i) = \lambda_i/\lambda$ for $i=1,2,\ldots,N$, $\lambda = \sum_{i=1}^N \lambda_i$

Then, $\forall i=1,2,\ldots,N$

$$\begin{aligned} \mathbb{E}V_i &= \mathbb{E}\sum_{n=0}^{\infty} 1(Z_n = i) = \sum_{n=0}^{\infty} \mathbb{P}(Z_n = i) \\ &= \mathbb{P}(Z_0 = i) + \sum_{n=0}^{\infty} \overbrace{\mathbb{P}(Z_{n+1} = i)}^{\sum_{j=1}^N \mathbb{P}(Z_n = j, Z_{n+1} = i)} \\ &= \mathbb{P}(Z_0 = i) + \sum_{n=0}^{\infty}\sum_{j=1}^N \mathbb{P}(Z_n = j) p_{ji} \\ &= \mathbb{P}(Z_0 = i) + \sum_{j=1}^N p_{ji} \underbrace{\sum_{n=0}^{\infty} \mathbb{P}(Z_n = j)}_{\mathbb{E}V_j} \end{aligned}$$

i.e.
$$\lambda \mathbb{E}V_i = \lambda_i + \sum_{j=1}^N (\lambda \mathbb{E}V_j) p_{ji}$$

Thus, setting $\bar{\lambda}_i = \lambda\mathbb{E}V_i$, we see they satisfy (*).

Uniqueness: (ES3)

APPLIED PROBABILITY
LECTURE 16

Jackson's Networks:
(James Jackson, 1957)

A network of N single-server queues with arrivals ind. P.P.(λ_k) and service dist ind. $\mathrm{Exp}(\mu_k)$ for $k = 1, 2, \dots, N$.
After service, each customer in queue i moves to queue j w.p. p_{ij} or exits the system w.p. $p_{i0} = 1 - \sum_j p_{ij}$.
We assume $p_{i0} > 0$ $\forall i$, $p_{ii} = 0$ $\forall i$, also the system is irred. Consider the queue lengths $X = (X_1, X_2, \dots, X_N)$. $I = \{0, 1, 2, \dots\}$.
If $n = (n_1, \dots, n_N) \in I$ and $e_i = (0, \dots, \underset{\uparrow i}{1}, \dots, 0)$.

Then
$$q_{n,n+e_i} = \lambda_i$$
$$\left.\begin{aligned} q_{n,n-e_i} &= \mu_i p_{i0} \\ q_{n,n-e_i+e_j} &= \mu_i p_{ij} \end{aligned}\right\} n_i \geq 1.$$

We say $\bar\lambda = (\bar\lambda_1, \dots, \bar\lambda_N)$ satisfies the traffic eq. if $\bar\lambda_i = \lambda_i + \sum_{j \neq i} \bar\lambda_j p_{ji}$. — $(*)$

Thm (Jackson): Assume that the soln of $(*)$ $\bar\lambda_i$ satisfy $\bar\lambda_i < \mu_i$ $\forall i = 1, \dots, N$. Then the J.N. is pos. recurrent with inv. distribution
$$\pi(n) = \prod_{i=1}^{N} (1 - \bar\rho_i) \bar\rho_i^{\,n_i}, \text{ where } \bar\rho_i = \frac{\bar\lambda_i}{\mu_i}.$$

At equilibrium, the <u>departure process</u> (to outside) are indep. P.P. with rates $\bar\lambda_i p_{i0}$.

The equilibrium is non-reversible, but there is "partial reversibility".

Lemma: (Partial detailed balance): Let X be a Markov process on I and π be a measure on I. Assume that for each $x \in I$, there is a partition of $I \setminus \{x\}$
$= I_1^x \cup I_2^x \cup \dots$ s.t. $\forall i \geq 1$,
$$\sum_{y \in I_i^x} \pi(x) q_{xy} = \sum_{y \in I_i^x} \pi(y) q_{yx}$$
then π is invariant for X, i.e. $\pi Q = 0$.
Rk: Of course, the detailed balance eqns satisfy the P.D.B. with singletons as partitions.

Proof of lemma:
$$(\pi Q)_y = \sum_x \pi(x) q_{xy} = \sum_{x \in I \setminus \{y\}} \pi(x) q_{xy} + \pi(y) q_{yy}$$
So,
$$(\pi Q)_y = \pi(y) q_{yy} + \sum_i \sum_{x \in I_i^y} \pi(x) q_{xy}$$
$$\stackrel{\text{P.D.B.}}{=} \pi(y) q_{yy} + \sum_i \sum_{x \in I_i^y} \pi(y) q_{yx}$$
$$= \pi(y) q_{yy} + \pi(y) \sum_i \sum_{x \in I_i^y} q_{yx}$$
$$= \pi(y) q_{yy} + \pi(y) \sum_{x \neq y} q_{yx} = 0.$$

Proof of thm (Jackson): Let
$$\pi(n) = \prod_{i=1}^{N} \bar\rho_i^{\,n_i}.$$ To show π is invariant.

Will show π satisfies the P.D.B.
Then for $\bar\rho_i < 1$, $\pi(n) = \prod (1 - \bar\rho_i) \bar\rho_i^{\,n_i}$ is inv. distribution.
As the rates are bounded, X is non-expl., hence pos. recurrent.

Let
$$A = \{e_i : 1 \leq i \leq N\},$$
$$D_j = \{-e_j + e_i : 1 \leq i \leq N, i \neq j\} \cup \{-e_j\}$$
for $1 \leq j \leq N$.

I.e., when a customer arrives and $n \in I$, $n \to n + m$ for $m \in A$.
when a customer leaves queue j & $n \in I$, $n \to n + m$, for $m \in D_j$.

Will show
$$(A) \quad \sum_{m \in A} \pi(n) q_{n,n+m} = \sum_{m \in A} \pi(n+m) q_{n+m,n}$$
$$(D) \quad \sum_{m \in D_j} \pi(n) q_{n,n+m} = \sum_{m \in D_j} \pi(n+m) q_{n+m,n}$$
$$\forall j = 1, \dots, N.$$

($\forall n \in I$, partition $I \setminus \{n\} = n + A \cup \bigcup_j n + D_j \cup G$.
$q_{n,m} = q_{m,n} = 0$ on G.)
& π satisfies the P.D.B. $\Rightarrow$ π inv.

D: fix j. To show $\sum_{m \in D_j} q_{n,n+m} = \sum_{m \in D_j} \frac{\pi(n+m)}{\pi(n)} q_{n+m,n}$

$$= \sum_{m \in D_j} q_{n,n+m} = \mu_j p_{j0} + \sum_{i=1}^{N} \mu_j p_{ji} = \mu_j (\underbrace{p_{j0} + \sum p_{ji}}_{1}) = \mu_j$$

$$\sum_{m \in D_j} \frac{\pi(n+m)}{\pi(n)} q_{n+m,n} = \frac{\pi(n-e_j)}{\pi(n)} q_{n-e_j,n} + \sum_{i \neq j} \frac{\pi(n-e_j+e_i)}{\pi(n)} \underbrace{q_{n-e_j+e_i,n}}_{m,\, m+e_j-e_i}$$
$$= \frac{1}{\bar\rho_j} \lambda_j + \sum_{i \neq j} \frac{\bar\rho_i}{\bar\rho_j} \mu_i p_{ij}$$
$$= \frac{\lambda_j}{\bar\rho_j} + \sum_{i \neq j} \frac{\bar\lambda_i}{\bar\rho_j} p_{ij} = \frac{1}{\bar\rho_j} (\underbrace{\lambda_j + \sum_{j \neq i} \bar\lambda_i p_{ij}}_{(*)})$$
$$= \frac{1}{\bar\rho_j} \bar\lambda_j = \mu_j.$$

(A)
$$\sum_{m \in A} q_{n,n+m} = \sum \lambda_i$$
$$\sum_{m \in A} \frac{\pi(n+m)}{\pi(n)} q_{n+m,n} = \sum_{m \in A} \frac{\pi(n+e_i)}{\pi(n)} q_{n+e_i,n}$$
$$= \sum_{m \in A} \bar\rho_i \mu_i p_{i0}.$$
$$= \sum_{m \in A} \bar\lambda_i p_{i0} = \bar\lambda_i (1 - \sum_j p_{ij})$$
$$= \sum_i \bar\lambda_i - \underbrace{\sum_i \bar\lambda_i \sum_j p_{ij}}_{\sum_j (\bar\lambda_j - \lambda_j)}$$
$$= \sum_j \lambda_j$$

The claim about departure processes is Ex. in ES3. $\square$

M/G/1 queue: Arrival: P.P.(λ), single-server queue, service time of nth customer is $\xi_n \geq 0$ and $(\xi_n)_{n \geq 1}$ are iid. and $E[\xi_1] = \frac{1}{\mu}$.

$(X_t)_{t \geq 0}$, the queue length, is no longer a MC. Let D_n be the departure time of the n^{th} customer. Then $(X_{D_n})_n$ is a discrete-time MC.

Prop: $Z_n = X_{D_n}$, $n = 0, 1, \dots$, $D_0 = 0$. Then Z_n has transition prob.
$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ p_0 & p_1 & p_2 & \cdots \\ 0 & p_0 & p_1 & \cdots \\ 0 & 0 & p_0 & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix} \text{ where}$$
$$p_k = \mathbb{E}\left[e^{-\lambda \xi_1} \frac{(\lambda \xi_1)^k}{k!} \right]$$

APPLIED PROBABILITY
LECTURE 17

M/G/1 Queue: Arrivals P.P.(λ)
Service times iid $(\xi_n)_{n\geq 0}$, with $\xi_n \geq 0$ and $\mathbb{E}\xi_1 = \frac{1}{\mu}$.

$(X_t)_{t\geq 0}$ is the queue length, let D_n be the departure time of the n^{th} customer, then $(X_{D_n})_n$ is a discrete-time MC.

Prop: $(Z_n)_n = (X_{D_n})_n$ is a MC with transition prob.

$$\begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ p_0 & p_1 & p_2 & \cdots \\ 0 & p_0 & p_1 & \cdots \\ 0 & 0 & p_0 & \cdots \\ & \vdots & & \ddots \end{pmatrix}$$

where $p_k = \mathbb{E}\left[e^{-\lambda\xi_1}\frac{(\lambda\xi_1)^k}{k!}\right]$

Proof: Let A_{n+1} be the # customers arriving after D_n and during the service-time of the $(n+1)$-st customer.

Then (A_n) are indep. (by the indep. increment ppty of P.P.) and given ξ_n, $A_n \sim \mathrm{Po}(\lambda\xi_n)$, i.e.
$$\mathbb{P}(A_n = k) = \mathbb{E}(\mathbb{P}(A_n = k \mid \xi_n)) = \mathbb{E}\left[e^{-\lambda\xi_n}\frac{(\lambda\xi_n)^k}{k!}\right] = p_k.$$

Now, $X(D_{n+1}) = \begin{cases} A_{n+1} & \text{if } X(D_n) = 0 \\ A_{n+1} + X(D_n) - 1 & \text{if } X(D_n) > 0 \end{cases}$

This gives the transition prob. as stated. $\square$

Thm: Let $\rho = \lambda/\mu$. If $\rho \leq 1$, the queue is recurrent in the sense that the queue empties out a.s.
If $\rho > 1$, then it is transient, in the sense that there is a positive probability it will never empty out.

Lemma: Let (Y_i) be iid $\mathbb{Z}$-valued random variables and let $S_n = Y_1 + \dots + Y_n$ be the corresponding rw starting from 0. If $\mathbb{E}[|Y_1|] < \infty$, then S is recurrent iff $\mathbb{E}[Y_1] = 0$.

Proof of Thm:
Proof 1: X is transient/recurrent (in the sense of the thm) iff (X_{D_n}) is transient/recurrent as a discrete-time MC.

While $X(D_n) > 0$, $X(D_n)$ is a random walk on $\mathbb{Z}^+$ with step distribution $Y_i = A_i - 1$. Also,
$$\mathbb{E}[Y_1] = \mathbb{E}[A_1] - 1 = \mathbb{E}[\mathbb{E}[A_1 \mid \xi_1]] - 1 = \mathbb{E}[\lambda\xi_1] - 1 = \lambda/\mu - 1 = \rho - 1.$$

Hence, if $\rho = 1$, then X is recurrent by previous lemma. If $\rho < 1$, it is a biased RW with bias towards 0, so recurrent $\}$ by SLLN.
If $\rho > 1$, ... ∞, so transient $\}$

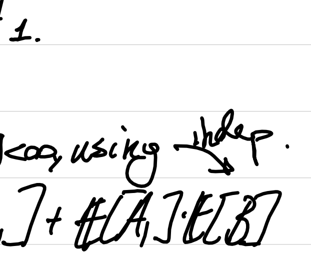
Proof 2: We will use a hidden branching structure. A customer C_2 is an offspring of C_1 if C_2 arrives during the service-time of C_1. This defines a tree.

The offspring-dist. is iid and distributed as A_1. This is a branching process. Recurrence of queue $\iff$ the branching process is finite a.s., iff $\mathbb{E}[A_1] \leq 1$ (recall from Prob IA).
$$\text{iff } \lambda/\mu = \rho \leq 1$$
$\square$

Def: (Busy period): Time between a customer joining an empty queue and a customer departing, leaving behind an empty queue is called a <u>busy period</u>.

Prop: For the M/G/1 queue, with $\lambda < \mu$, the length of the busy period B satisfies
$$\mathbb{E}B = \frac{1}{\mu - \lambda}$$

Intuition
$\mathbb{E}[\text{size of tree}] = \frac{1}{1-\rho}$
$\mathbb{E}[\text{service time for each customer}] = 1/\mu$
So $\mathbb{E}[\text{Busy}] = \frac{1}{1-\rho}\cdot\frac{1}{\mu} = \frac{1}{\mu - \lambda}$



$B = \xi_1 + \sum_{i=1}^{A_1} B_i$, where B_i's are iid copies of B and the $(B_i)_i \perp\!\!\!\perp A_1$.
A_1 & ξ_1 not indep.

Taking expectations, (assuming $\mathbb{E}[B] < \infty$, using indep.) $\mathbb{E}[B] = \mathbb{E}[\xi_1] + \mathbb{E}[A_1]\mathbb{E}[B]$ (left as an exercise).

Proof of lemma: By SLLN, if $\mathbb{E}[Y_1] > 0$, then $S_n \to \infty$ a.s. Similarly, if $\mathbb{E}[Y_1] < 0$, $S_n \to -\infty$ a.s. hence transient.

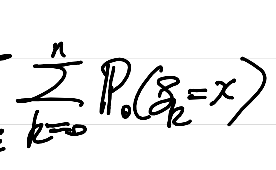
Assume $\mathbb{E}[Y_1] = 0$. By SLLN, $\frac{S_n}{n} \to 0$ a.s. & in $\mathbb{P}$.

Fix $\varepsilon > 0$, then for some n large enough,
$$\min_{1\leq i\leq n}\mathbb{P}(|S_i| < \varepsilon n) \geq \frac{1}{2} \qquad (*)$$

Choose N_1 large enough s.t. $\mathbb{P}(|S_n| < \varepsilon n) \geq 1/2$ $\forall n \geq N_1$.
Now, get $N_2 > N_1$ s.t. $\mathbb{P}(|S_i| < \varepsilon N_2) \geq \frac{1}{2}$, $\forall i = 1, 2, \dots, N_1$.
Then $(*)$ holds for $n = N_2$.

Let
$$G_n(x) = \mathbb{E}_0[\# \text{ visits to } x, \text{ until time } n] = \mathbb{E}_0\left[\sum_{k=0}^{n}\mathbb{1}(S_k = x)\right] \overset{MCT}{=} \sum_{k=0}^{n}\mathbb{P}_0(S_k = x).$$

Clearly, $G_n(x)$ is increasing in n. ✓
Also, $\forall x$, $G_n(x) \leq G_n(0)$, because,
$$G_n(x) \overset{(=)}{=} \sum_{k=0}^{n}\mathbb{P}_0(T_x = k)\cdot G_{n-k}(0)$$



$T_x = \inf\{j \geq 0 : S_j = x\}$ $\Big)$ $G_n(0)$ inc. in $n \in \mathbb{N}$
$$\overset{(\leq)}{\leq} \sum_{k=0}^{n}\mathbb{P}_0(T_x = k)\cdot G_n(0) = G_n(0)$$

Thus, with the n in $(*)$,
$$\underbrace{2n\varepsilon G_n(0)} \geq \sum_{|x|<n\varepsilon} G_n(x) = \sum_{|x|<n\varepsilon}\sum_{k=0}^{n}\mathbb{P}_0(S_k = x)$$
$$= \sum_{k=0}^{n}\sum_{|x|<n\varepsilon}\mathbb{P}_0(S_k = x)$$
$$= \sum_{k=0}^{n}\underbrace{\mathbb{P}_0(|S_k| < n\varepsilon)}_{\geq 1/2 \text{ by } (*)}$$
$$\geq n/2.$$

So, $G_n(0) \geq \frac{1}{4\varepsilon}$, i.e.,
$$\mathbb{E}_0 V_0 \geq \frac{1}{4\varepsilon}$$
V_0: # visits to 0.

As $\varepsilon > 0$ was arbitrary, this shows $\mathbb{E}_0 V_0 = \infty$, and so $(S_n)_n$ is recurrent. $\square$

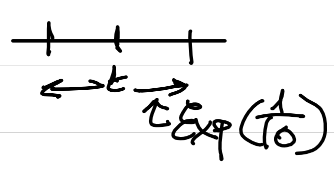
Renewal Processes:

Suppose buses arrive "every 10 minutes" according to the following two models ——
(a) Buses arrive exactly after 10 minutes from the previous one
(b) Appear according to a P.P.($1/10$), i.e. next bus arrives after an indep. Exp with mean 10 mins.

How long do I have to wait on average?

(a) 5 minutes
(b) 10 minutes

What is the inter-arrival time containing t in (b)?
No longer an Exp($1/10$), but longer.



What happens when the n^{th} bus arrives after time ξ_n from the $(n-1)^{th}$ bus, $\xi_n \geq 0$, $(\xi_n)_n$ iid. The length of the interarrival interval containing t is longer than ξ_1, in fact, for t large, this is the "size-biased" distribution of ξ_1.

Suppose you want to estimate the avg. family size in Cambridge.

$$\frac{X_1+\dots+X_n}{n} = \frac{Y_1^2+\dots+Y_k^2}{Y_1+\dots+Y_k} \rightarrow \frac{E[Y_1^2]}{E[Y_1]}$$

Exp of size-biased dist. of Y_1.

Def: Let $(\xi_i)_{i\geq 1}$ be iid non-neg r.v.'s with distr. same as ξ with $\mathbb{P}(\xi>0)>0$. Set $T_n = \sum_{i=1}^{n} \xi_i$ and $N_t = \max\{n\geq 0 : T_n \leq t\}$ is called the renewal process and ξ_n is the time of the renewal.

Rk: If ξ_i are iid Exp(λ), N_t is P.P.(λ).

Thm: If $E[\xi] = \frac{1}{\lambda} < \infty$, then as $t\to\infty$,
$\frac{N_t}{t} \to \lambda$ a.s., $\frac{\mathbb{E}N_t}{t} \to \lambda$.

Proof: We will only prove the first claim. See Grimmett - Stirzaker for proof of the second claim.

First note that, $N_t < \infty$ a.s. for any $t>0$, and $N_t \to \infty$, as $t\to\infty$.
Also, $T_{N_t} \leq t < T_{N_t+1}$ by defn of N_t.

$$\Rightarrow \quad \frac{T_{N_t}}{N_t} \leq \frac{t}{N_t} < \frac{T_{N_t+1}}{N_t}$$

By SLLN, $\frac{T_n}{n} \to \frac{1}{\lambda}$ a.s. and $N_t\to\infty$ a.s., so

$$\frac{T_{N_t}}{N_t} \to \frac{1}{\lambda} \text{ a.s. as } t\to\infty.$$

$$\& \quad \frac{T_{N_t+1}}{N_t+1}\left(1+\frac{1}{N_t}\right) \longrightarrow \frac{1}{\lambda} \text{ a.s.}$$

Thus, $\frac{t}{N_t} \to \frac{1}{\lambda}$ a.s., i.e. $\frac{N_t}{t} \to \lambda$ a.s. $\square$

Size-biased Picking: Now suppose $\mathbb{P}(\xi>0)=1$. Let $S_i = \xi_1+\dots+\xi_i$, $1\leq i\leq n$.
Use $\frac{S_i}{S_n}$, $1\leq i\leq n$ to partition $[0,1]$ into n sub-intervals of size $Y_i = \frac{\xi_i}{S_n}$, $i=1,\dots,n$.

$0 \quad \frac{S_1}{S_n} \quad \frac{S_2}{S_n} \quad \dots \quad \frac{S_{n-1}}{S_n} \quad 1=\frac{S_n}{S_n}$

(Y_i) are identically distributed. Let $U\sim$Unif$[0,1]$ indep. of $(\xi_i)_{i\geq 1}$, and let $\hat{Y}$ denote the length of the interval containing U. What is the dist of $\hat{Y}$?

Not same as Y_1 as U tends to fall in bigger intervals.

This is called the size-biased picking.

Prop: $\mathbb{P}(\hat{Y}\in dy) = ny\,\mathbb{P}(Y_1\in dy)$.
(in particular, if Y_1 has a density fn f_{Y_1}, then $\hat{Y}$ has the density function
$$f_{\hat{Y}}(y) = ny f_{Y_1}(y) \quad).$$

Proof:
$$\begin{aligned}
\mathbb{P}(\hat{Y}\in dy) &= \sum_{i=1}^{n} \mathbb{P}\left(\hat{Y}\in dy, \frac{S_{i-1}}{S_n} \leq U \leq \frac{S_i}{S_n}\right)\\
&= \sum_{i=1}^{n} \mathbb{P}\left(\frac{\xi_i}{S_n}\in dy, \frac{S_{i-1}}{S_n} \leq U \leq \frac{S_i}{S_n}\right)\\
&= \sum_{i=1}^{n} \mathbb{P}\left(\frac{S_{i-1}}{S_n}\leq U \leq \frac{S_i}{S_n} \,\middle|\, \frac{\xi_i}{S_n}=y\right)\mathbb{P}\left(\frac{\xi_i}{S_n}\in dy\right)\\
&= \sum_{i=1}^{n} y\,\mathbb{P}\left(\frac{\xi_i}{S_n}\in dy\right) = ny\,\mathbb{P}(Y_1\in dy)
\end{aligned}$$
$\square$

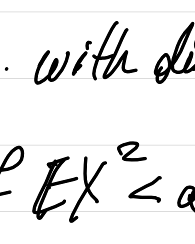
Def: Let X be a non-neg r.v. with distribution μ and $\mathbb{E}[X]<\infty$. Then, the size-biased d^{n} of μ is $\hat{\mu}(dx) = \frac{1}{\mathbb{E}[X]} x\mu(dx)$.

If X has a density f^n f_μ, then the size-biased d^n of X also has a density f^n
$f_{\hat{\mu}}(x) = \frac{x}{m} f(x)$.

We write $\hat{X}$ for the r.v. with dist. $\hat{\mu}$.

Rk: $\mathbb{E}\hat{X} = \frac{\mathbb{E}X^2}{\mathbb{E}X} \geq \mathbb{E}[X]$ if $\mathbb{E}X^2<\infty$.

Ex: If $X\sim U[0,1]$, then $\hat{X}$ has density $2x$ on $[0,1]$

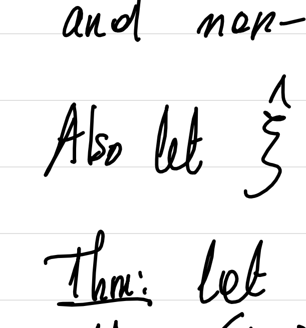


Ex: $X\sim$ Exp(λ), then $\hat{X}$ has density
$$\lambda x \lambda e^{-\lambda x} = \lambda^2 x e^{-\lambda x} \text{ for } x\geq 0,$$
i.e. $\hat{X}\sim$ Gamma$(2,\lambda)$, i.e. $\hat{X}$ has the same dist^n as X_1+X_2, where $X_1, X_2 \overset{iid}{\sim}$ Exp(λ).

Equilibrium Theory for Renewal Processes:

Q: For large t, how long do I have to wait for the next renewal?

Given a renewal process $(X_t)_{t\geq 0}$ and a fixed $t>0$, set



$A(t) = t - T_{N_t}$, age process
$E(t) = T_{N_t+1} - t$: excess arrival / residual life
$L(t) = A(t)+E(t)$
$\quad = T_{N_t+1} - T_{N_t}$, length of current renewal

Def: A random variable ξ is called arithmetic if $\mathbb{P}(\xi\in k\mathbb{Z})=1$ for some $k>1$, $k\in\mathbb{Z}$ and non-arithmetic if $\forall k>1$, $k\in\mathbb{Z}$ $\mathbb{P}(\xi\in k\mathbb{Z})<1$.

Also let $\hat{\xi}$ be the size-biased dist. of ξ.

Thm: Let ξ be non-arithmetic. Let $\mathbb{E}\xi = \frac{1}{\lambda}$.
Then, $(L(t), E(t)) \xrightarrow[t\to\infty]{d} (\hat{\xi}, U\hat{\xi})$ where

$U\sim$ Unif$[0,1]$ indep. of $\hat{\xi}$.
Similarly, $(L(t), A(t)) \xrightarrow[t\to\infty]{d} (\hat{\xi}, U\hat{\xi})$

where $U\sim$ Unif$[0,1]$ indep. of $\hat{\xi}_1$.

i.e. $\mathbb{P}(L(t)\leq x, E(t)\leq y) \longrightarrow \mathbb{P}(\hat{\xi}\leq x, U\hat{\xi}\leq y)$
$\forall x,y$ continuity pts of $(\hat{\xi}, U\hat{\xi})$.

Rk 1: $L(t)$ for large t has the size-biased dist. $\hat{\xi}$, and given $L(t)$, the point t falls uniformly within the current renewal interval of length $L(t)$.

Rk 2: $\mathbb{P}(E(t)\leq y) \xrightarrow{t\to\infty} \mathbb{P}(U\hat{\xi}\leq y)$
$$= \lambda\int_0^y \mathbb{P}(\xi>z)\,dz.$$

proof in next class

APPLIED PROBABILITY
LECTURE 19

Renewal Process:

Proof of ⊛:

$$\mathbb{P}(U\hat{\xi}\le y) = \int_0^1 \mathbb{P}(\hat{\xi}\le y/u)\,du = \int_0^1\int_0^{y/u}\mathbb{P}(\hat{\xi}\in dz)\,du$$

$$\begin{cases}0\le z\le y/u\\ 0\le u\le 1\end{cases} \Updownarrow \begin{cases}0\le z\le\infty\\ 0\le u\le\min(1,\frac{y}{z})\end{cases}$$

$$= \int_0^1\int_0^{y/u}\lambda z\,\mathbb{P}(\xi\in dz)\,du = \int_0^\infty\lambda\int_0^{1\wedge y/z}du\,\mathbb{P}(\xi\in dz) = \underline{\int_0^\infty \lambda\left(\tfrac{y}{x}\wedge 1\right)\mathbb{P}(\xi\in dx)}$$

$$\lambda\int_0^y \mathbb{P}(\xi>z)\,dz = \int_0^y\lambda\int_z^\infty \mathbb{P}(\xi\in dx)\,dz$$

$$\begin{cases}z\le x<\infty\\ 0\le z\le y\end{cases}\Updownarrow\begin{cases}0\le z\le\min(x,y)\\ 0\le x<\infty\end{cases}$$

$$= \lambda\int_0^\infty\int_0^{x\wedge y}dz\,\mathbb{P}(\xi\in dx) = \int_0^\infty\lambda(x\wedge y)\,\mathbb{P}(\xi\in dx)$$

Hence $\mathbb{P}(U\hat{\xi}\le y) = \lambda\int_0^y \mathbb{P}(\xi>z)\,dz$

Example 1: If $\xi\sim Exp(\lambda)$, then $\hat{\xi}\sim Gamma(2,\lambda)$ and $\mathbb{P}(U\hat{\xi}\le y) = \lambda\int_0^y \mathbb{P}(\xi>z)\,dz$

$$= \lambda\int_0^y e^{-\lambda z}\,dy = 1-e^{-\lambda y}$$

$U\hat{\xi}\sim Exp(\lambda)$

Thus, $E(t)\xrightarrow{d} Exp(\lambda)$: $E(t)\sim Exp(\lambda)\ \forall t$

$A(t)\xrightarrow[t\to\infty]{d} Exp(\lambda)$: $A(t)\sim$

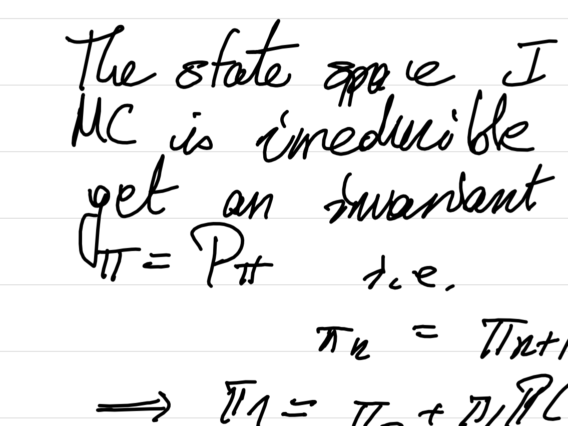
Ex. $\begin{cases}\text{To find } A(t)\text{, show } A(t)\,\&\,E(t)\text{ are indep. Hence}\\ \text{prove that } L(t)\xrightarrow{d} Gamma(2,\lambda).\end{cases}$

Example 2: $\xi\sim U(0,1)$. Then for $0\le y\le 1$, $E(t)\xrightarrow{d} E_\infty$, then $\mathbb{P}(E_\infty\le y) = \lambda\int_0^y \mathbb{P}(\xi>z)\,dz$

$$= 2\int_0^y (1-z)\,dz \qquad \lambda = \frac{1}{\mathbb{E}\xi} = 2.$$

$$= 2\left(y-\frac{y^2}{2}\right).$$

Proof of Thm: (only for ξ discrete, i.e., ξ takes values in $\{1,2,\dots\}$, and time is discrete



$(E(t): t=0,1,2,\dots)$ is a discrete-time MC. with transition prob. $p_{i,i-1}=1$, $\forall i\ge 2$ and $p_{1,n} = \mathbb{P}(\xi=n)\ \forall n\ge 1$.

The state space $I=\{1,2,\dots\}$, and the MC is irreducible and recurrent and to get an invariant distribution π, solve $\pi = P\pi$ i.e.

$$\pi_n = \pi_{n+1} + \pi_1\mathbb{P}(\xi=n)\quad\forall n\ge 1.$$

$$\Longrightarrow \pi_1 = \pi_2 + \pi_1\mathbb{P}(\xi=n),\ \text{so } \pi_2 = \pi_1\mathbb{P}(\xi\ge 2),$$
$$\pi_2 = \pi_3 + \pi_2\mathbb{P}(\xi=2),\ \text{so } \pi_3 = \pi_1\mathbb{P}(\xi\ge 3).$$

By induction, $\pi_n = \pi_1\mathbb{P}(\xi\ge n)$.

Since $\sum \pi_1\mathbb{P}(\xi\ge n) = \pi_1\mathbb{E}\xi = \frac{\pi_1}{\lambda}$, and so,

$$\pi_n = \lambda\mathbb{P}(\xi\ge n).$$

As ξ is non-arithmetic, $(E(t))$ is aperiodic, hence, $\mathbb{P}(E(t)=k)\xrightarrow[t\to\infty]{}\pi_k$.

Thus, for y integer,

$$\mathbb{P}(E(t)\le y)\longrightarrow\sum_{i=1}^{y}\pi_i = \lambda\int_0^y \mathbb{P}(\xi>z)\,dz$$

Now for $(L(t),E(t))_{t=0,1,2,\dots}$, this will also be a discrete-time MC with state space $I=\{(n,k): \underset{n=1,2,\dots}{1\le k\le n}\}\subset\mathbb{N}\times\mathbb{N}$

With transition probabilities

$$\begin{cases}P_{(n,k)\to(n,k-1)} = 1, & \text{when } k\ge 2\\ P_{(n,1)\to(m,m)} = \mathbb{P}(\xi=m), & (n,m)\in\mathbb{N}\times\mathbb{N}.\end{cases}$$

this again is aperiodic, irred, recurrent, with invariant measure, $\pi=\pi P$

$\pi(n,k-1) = \pi(n,k)$ for $2\le k\le n$, and

$$\pi(m,m) = \sum_n \pi(n,1)\mathbb{P}(\xi=m)$$
$$= C\mathbb{P}(\xi=m)$$

and hence $\pi(n,k) = C\mathbb{P}(\xi=n)$

Also, $1 = \sum_n\sum_{k=1}^n \pi(n,k) = \sum_n\sum_{k=1}^n C\mathbb{P}(\xi=n)$

$$= \sum_n Cn\,\mathbb{P}(\xi=n)$$

i.e., $C=\lambda$ $\qquad = C\mathbb{E}\xi = \frac{C}{\lambda}$

I.e., $\pi(n,k) = \lambda\mathbb{P}(\xi=n)$

$$= \underbrace{\lambda n\mathbb{P}(\xi=n)}_{\mathbb{P}(\hat{\xi}=n)}\cdot\underbrace{\tfrac{1}{n}\mathbb{1}(1\le k\le n)}_{\substack{\text{Given } L_\infty=n\\ E_\infty\ Unif\{1,\dots,n\}}}.$$

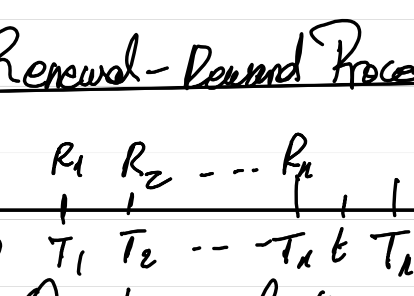
and $\mathbb{P}(L(t)\le x, E(t)\le y)\longrightarrow\sum_{n\le x,k\le y}\pi(n,k)$.

$\square$

Rk: In fact, can show for any fixed $t>0$, $\mathbb{P}(L(t)\ge x)\ge\mathbb{P}(\xi\ge x)\ \forall x.$ — ⊛

I.e. $L(t)$ is "stochastically larger" than ξ

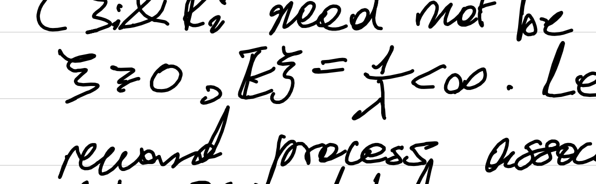
(For two r.v.'s X,Y, X is stochastically larger than Y if $\mathbb{P}(X\ge z)\ge\mathbb{P}(Y\ge z)\ \forall z$).



Hence also, $\mathbb{E}X\ge\mathbb{E}Y$.

⊛ This is called the inspection paradox. (Exercise).

Renewal-Reward Process:



On top of the renewal structure, there is a reward associated to each renewal, which could be a function of the renewal. Let $(\xi_i,R_i)_{i\in\mathbb{N}}$ be iid pairs of r.v.'s ($\xi_i\,\&\,R_i$ need not be independent) with $\xi\ge 0$, $\mathbb{E}\xi=\frac{1}{\lambda}<\infty$. Let $(N_t)_{t\ge0}$ be the renewal process associated with (ξ_i) and let $R(t)$ = total reward until time t

$$= \sum_{i=1}^{N_t}R_i$$

Prop: If $\mathbb{E}|R|<\infty$, as $t\to\infty$

$$\frac{R_t}{t}\longrightarrow\frac{\mathbb{E}R}{\mathbb{E}\xi} = \lambda\mathbb{E}R\ \text{a.s. and}$$

$$\frac{\mathbb{E}R_t}{t}\longrightarrow\lambda\mathbb{E}R.$$

Proof: $\frac{R(t)}{t} = \frac{\sum_{i=1}^{N_t}R_i}{N_t}\cdot\frac{N_t}{t}$. (Complete from here as before). $\square$

In fact, for the current renewal $r(t) = \mathbb{E}(R_{N_t+1})$.

Thm: As $t\to\infty$, $r(t)\longrightarrow\lambda\mathbb{E}(R\xi)$.

Proof: Skipped.

Rk: the factor ξ comes from size-biasing, the reward R has been biased by the size of the current renewal.

Example: Alternating renewal process: A machine breaks after time X_i, then it takes time Y_i to get fixed $(X_1, Y_1 \geq 0)$ iid, indep. of each other. Thus, $\xi_i = X_i + Y_i$ is the length of a full cycle and defines a renewal process. What is the fraction of time the machine runs in the long run?

Associate to each renewal the reward R_i, corresponding to the amount of time the machine was running in cycle i, i.e. $R_i = X_i$. Then (ξ_i, R_i) is a renewal-reward process, and $R(t)$ is total time till t the machine was running.

$$\frac{R(t)}{t} \xrightarrow{t\to\infty} \frac{\mathbb{E}R}{\mathbb{E}\xi} = \frac{\mathbb{E}X}{\mathbb{E}X+\mathbb{E}Y} \text{ a.s. and in expectation}$$

$R(t)/t$ is not exactly the fraction of time that the machine was running, if we define $R(t)$ as in the renewal-reward thm. But the maximal difference is R_{N_t+1} & $\frac{R_{N_t+1}}{t} \longrightarrow 0$ a.s. and in expectation.

Example: Optimal replacement strategy: A car has a random life V. The replacement cost is C_1 if the car has not failed yet and $C_1+C_2 > C_1$ if the car fails. What is the optimal schedule for replacing the car to minimise the long-term running cost?

Renewal reward process strategy: We buy a new car & then after a fixed time T we buy a new car again if it hasn't failed till time T.

Renewal: $\xi_i = \min(V_i, T)$
Reward: $R_i = C_1 + C_2 \mathbb{1}(V_i < T) \to$ cost for renewal.

For large t, $R(t) \sim \frac{t\,\mathbb{E}R}{\mathbb{E}\xi}$. So we need to minimise $\mathbb{E}R/\mathbb{E}\xi$.

Now, $\mathbb{E}R = C_1 + C_2\,\mathbb{P}(V<T) = C_1 + C_2 F(T)$
where F is the cdf of V.
$\mathbb{E}\xi = \int_0^\infty \mathbb{P}(\xi > x)\,dx = \int_0^T \mathbb{P}(V>x)\,dx = \int_0^T \bar{F}(x)\,dx$

$(\bar{F}(\cdot) = 1 - F(\cdot))$

Thus, to minimise $g(T) = \dfrac{C_1 + C_2 F(T)}{\int_0^T \bar{F}(x)\,dx}$

E.g.: Take $V \sim \text{Unif}(0,2)$. Then, $(0 \leq T \leq 2)$
$F(T) = T/2$, $\int_0^T \bar{F}(x)\,dx = \int_0^T 1 - x/2\,dx = T - T^2/4$.
So $g(T) = \dfrac{4C_1 + 2C_2 T}{4T - T^2}$

$\Rightarrow$ $g(T)$ minimised at $T^* = 2(\sqrt{a^2+2a} - a) < 2$
where $a = C_1/C_2$.

Example: (Train dispatching): Suppose that passengers arrive to a platform according to a renewal process at rate μ. As soon as N passengers arrive, a train departs with all N on board. The process continues. The company incurs a cost at the rate of nc per unit time when n passengers are waiting and a fixed cost K every time a train departs.
What is the optimal value of N?

Proof: Exc.

Little's formula: (John Little, 1961):

Def: A process $(X_t : t \geq 0)$ is regenerative if there exist random times τ_n s.t. the process regenerates, i.e. $(X_{t+\tau_n}, t \geq 0) \overset{d}{=} (X_t : t \geq 0)$ and $(X_{t+\tau_n}, t \geq 0)$ is indep. of $(X_t : t \leq \tau_n)$.
Also, $\tau_0 = 0$, $\tau_{n+1} \geq \tau_n$ and τ_{n+1} depends only on $(X_{t+\tau_n} : t \geq 0)$.
In particular, $(\tau_{n+1} - \tau_n)_{n\geq 0}$ are iid.

Rk: An M/G/1 queue is regenerative with τ_n as the end of the n^{th} busy period.

Thm: Let X be a queue starting from 0 that is regenerative with regeneration times τ_n. Let N be the arrival process of X and let W_i be the waiting time of the i^{th} customer (including the service time).
Assume that $\mathbb{E}\tau_1 < \infty$ and $\mathbb{E}N_{\tau_1} < \infty$, then a.s. the following limits exist and are deterministic —

(a) Long-run mean queue size:
$L = \lim_{t\to\infty} \frac{1}{t}\int_0^t X_s\,ds$

(b) Long-run avg. wait time:
$W = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} W_i$

(c) Long-run avg. arrival rate:
$\lambda = \lim_{t\to\infty} \frac{N_t}{t}$

And $\boxed{L = \lambda W}$

Proof: Set $Y_n = \sum_{i=1}^{N_{\tau_n}} W_i$. Since $X_0 = 0$, by the regeneration property $X_{\tau_n} = 0$ $\forall n$.
Let $\tau_n \leq t \leq \tau_{n+1}$, then

(*) $\frac{1}{\tau_{n+1}} Y_n \leq \frac{1}{t}\int_0^t X_s\,ds \leq \frac{1}{\tau_n} Y_{n+1}$

$\int_0^t X_s\,ds \leq \int_0^{\tau_{n+1}} X_s\,ds = \sum_{i=1}^{N_{\tau_{n+1}}} W_i = Y_{n+1}$.

By the regeneration ppty $(Y_{n+1} - Y_n)_n$ are iid.
So $\frac{Y_n}{\tau_n} = \frac{Y_1 + Y_2 - Y_1 + \cdots + Y_n - Y_{n-1}}{n} \cdot \frac{n}{\tau_n - \tau_{n-1} + \cdots + \tau_1}$
$\xrightarrow{n\to\infty} \frac{\mathbb{E}Y_1}{\mathbb{E}\tau_1}$ by SLLN.
and $\frac{Y_{n+1}}{\tau_n}$

Hence from (*), as $t\to\infty$ $\frac{1}{t}\int_0^t X_s\,ds \to \frac{\mathbb{E}Y_1}{\mathbb{E}\tau_1} = L$ a.s.

Similarly, $N_{\tau_n} = \sum_{i=0}^{n-1} N_{(\tau_i, \tau_{i+1}]}$ $\hookrightarrow$ iid by regen. so,
$\frac{N_{\tau_n}}{\tau_{n+1}} \leq \frac{N_t}{t} \leq \frac{N_{\tau_{n+1}}}{\tau_n}$ and $\xrightarrow[\text{a.s.}]{n\to\infty}$ by SLLN λ

where $\lambda = \frac{\mathbb{E}N_{\tau_1}}{\mathbb{E}\tau_1}$

Also, for $N_{\tau_n} \leq k \leq N_{\tau_{n+1}}$,
$Y_n \frac{1}{N_{\tau_{n+1}}} \leq \frac{1}{k}\sum_{i=1}^{k} W_i \leq \frac{1}{N_{\tau_n}} Y_{n+1}$

And $\frac{Y_n}{N_{\tau_{n+1}}}$ and $\frac{Y_{n+1}}{N_{\tau_n}}$ converge a.s. to

$\left(\frac{Y_n}{N_{\tau_{n+1}}} = \frac{Y_n}{n} \cdot \frac{\tau_{n+1}}{N_{\tau_{n+1}}} \cdot \frac{n+1}{\tau_{n+1}} \cdot \frac{n}{n+1} \longrightarrow \mathbb{E}Y_1 \cdot \frac{1}{\lambda} \cdot \frac{1}{\mathbb{E}\tau_1}\right)$.

Rk: In fact, $L = \lambda W$ holds only assuming the limits in (b) & (c) exist and $X_t/t \to 0$ a.s.

Proof: $\sum_{i=1}^{N_t - X_t} W_i \leq \int_0^t X_s\,ds \leq \sum_{i=1}^{N_t} W_i$,

so $\frac{\sum_{i=1}^{N_t} W_i}{t} = \frac{\sum_{i=1}^{N_t} W_i}{N_t} \cdot \frac{N_t}{t} \longrightarrow W \cdot \lambda$ a.s. by (b) & (c).

$\frac{\sum_{i=1}^{N_t - X_t} W_i}{t} = \underbrace{\frac{\sum_{i=1}^{N_t - X_t} W_i}{N_t - X_t}}_{\to W} \cdot \frac{N_t - X_t}{t} \longrightarrow W \cdot \lambda$ a.s.
$\frac{N_t - X_t}{t} \to \lambda$, as $\frac{X_t}{t} \to 0$.

$\square$

Applied Probability
Lecture 21

Spatial Poisson Process:

I want to model the distribution of the location
- of a plant in a certain area.
- location of stars in space.

Definition & Superposition:

The standard Poisson Process on $\mathbb{R}_+$ can be encoded by the set of jump times $0 < J_1 < J_2 < \dots$

$$\Pi \stackrel{\text{def}}{=} \{J_1, J_2, \dots\} \subseteq [0, \infty).$$

Π is a random countable subset of $[0,\infty)$.
A spatial Poisson process is a random countable $\Pi \subseteq \mathbb{R}^d$.

Measure-theoretic facts: let $\tilde{\mathcal{B}}(\mathbb{R}^d) := \{\prod_{i=1}^d (a_i, b_i] : a_i < b_i\}$ be the set of boxes in $\mathbb{R}^d$. The volume is $|A| = \prod_{i=1}^d (b_i - a_i)$.

The "Borel σ-algebra" $\mathcal{B}(\mathbb{R}^d)$ is the smallest σ-algebra containing all elements of $\tilde{\mathcal{B}}(\mathbb{R}^d)$.
For $A \in \mathcal{B}(\mathbb{R}^d)$, the "volume" (Lebesgue measure) $|A|$ is still defined.

Elements in $\mathcal{B}(\mathbb{R}^d)$ are called Borel sets.
A function $f: \mathbb{R}^d \longrightarrow \mathbb{R}^s$ is called measurable if $f^{-1}B = \{x \in \mathbb{R}^d : f(x) \in B\} \in \mathcal{B}(\mathbb{R}^d) \ \forall B \in \mathcal{B}(\mathbb{R}^s)$.

Def: A random countable subset $\Pi \subseteq \mathbb{R}^d$ is a Poisson process with constant intensity $\lambda > 0$, if $\forall A \in \mathcal{B}(\mathbb{R}^d)$:
(a) $N(A) = \#(\Pi \cap A) \sim \mathrm{Poi}(\lambda |A|)$
(b) For any $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}^d)$ disjoint, $N(A_1), \dots, N(A_n)$ are independent.

If $|A| = \infty$ we interpret (a) as $N(A) = \infty$ a.s.

Rk: If Π is a spatial Poisson process on $\mathbb{R}_+$ with constant intensity $\lambda > 0$, then $N(t) = N([0,t]) \ \forall t$, $(N(t))_{t \geq 0}$ is a standard P.P.(λ) on $\mathbb{R}^+$.

Def: Let $\lambda : \mathbb{R}^d \to \mathbb{R}$, non-negative, measurable s.t. $\Lambda(A) = \int_A \lambda(x)\,dx < \infty$ $\forall$ bdd Borel A, then Π is a non-homogeneous P.P. with intensity function λ if $\forall A \in \mathcal{B}(\mathbb{R}^d)$,
(a) $N(A) = \#(\Pi \cap A) \sim \mathrm{Poi}(\Lambda(A))$
(b) For any $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}^d)$ disjoint, $N(A_1), \dots, N(A_n)$ are indep.

Λ is called the "mean measure" of the Poisson Process.

Rk: one can define a Poisson process with mean measure Λ directly, when $\Lambda(\{x\}) = 0 \ \forall x \in \mathbb{R}^d$.

Thm: (Superposition): Let Π_1, Π_2 be two indep. Poisson processes with intensity f^{n}s λ_1, λ_2 resp. Then $\Pi = \Pi_1 \cup \Pi_2$ is a P.P. with intensity f^n $\lambda = \lambda_1 + \lambda_2$.

Proof: Let $N_1(A) = \#(\Pi_1 \cap A)$ $\forall A \in \mathcal{B}(\mathbb{R}^d)$.
$N_2(A) = \#(\Pi_2 \cap A)$
Then, by definition, $N_i(A) \sim \mathrm{Poi}(\Lambda_i(A))$ where $\Lambda_i(A) = \int_A \lambda_i(x)\,dx$ $\forall i = 1, 2$ and $N_1(A), N_2(A)$ are indep.

Define $S(A) = N_1(A) + N_2(A)$ $\forall A \in \mathcal{B}(\mathbb{R}^d)$.
Then $S(A) \sim \mathrm{Poi}(\underbrace{\Lambda_1(A) + \Lambda_2(A)}_{:= \Lambda(A)})$, then

Also, if $A_1, A_2, \dots$ disjoint, then $S(A_1), S(A_2), \dots$ are indep.

Are we done?
Want: $S(A) = \#(\Pi \cap A) = \#(\Pi_1 \cap A) + \#(\Pi_2 \cap A) - \#(\Pi_1 \cap \Pi_2 \cap A)$
$= N_1(A) + N_2(A) - \#(\Pi_1 \cap \Pi_2 \cap A)$

I.e., need to show $\Pi_1 \cap \Pi_2 \cap A = \emptyset$ a.s. $\forall A \in \mathcal{B}(\mathbb{R}^d)$.
Enough to show this for all A bounded. Let

$$Q_{k,n} = \prod_{i=1}^d (k_i \cdot 2^{-n}, (k_i+1) \cdot 2^{-n}] \text{ for } k = (k_1, \dots, k_d) \in \mathbb{Z}^d, n \in \mathbb{N}$$

will call $Q_{k,n}$ an "n-box". For diff values of k, and a fixed n, $Q_{k,n}$'s are disjoint.

For any fixed $n \in \mathbb{N}$

$$\mathbb{P}(\Pi_1 \cap \Pi_2 \cap A \neq \emptyset) \leq \sum_{k \in \mathbb{Z}^d} \mathbb{P}(N_1(Q_{k,n} \cap A) \geq 1, N_2(Q_{k,n} \cap A) \geq 1)$$

$$= \sum_{k \in \mathbb{Z}^d} \mathbb{P}(N_1(Q_{k,n} \cap A) \geq 1) \cdot \mathbb{P}(N_2(Q_{k,n} \cap A) \geq 1)$$

$$= \sum_{k \in \mathbb{Z}^d} (1 - e^{-\Lambda_1(Q_{k,n} \cap A)}) \cdot (1 - e^{-\Lambda_2(Q_{k,n} \cap A)})$$

$1 - e^{-x} \leq x \ \forall x$

$$\leq \sum_{k \in \mathbb{Z}^d} \Lambda_1(Q_{k,n} \cap A) \cdot \Lambda_2(Q_{k,n} \cap A)$$

$$\leq \underbrace{\max_{k \in \mathbb{Z}^d} \Lambda_1(Q_{k,n} \cap A)}_{M_n(A)} \underbrace{\sum_{k \in \mathbb{Z}^d} \Lambda_2(Q_{k,n} \cap A)}_{\Lambda_2(A) < \infty \text{ as } A \text{ is bdd}}.$$

So, $\mathbb{P}(\Pi_1 \cap \Pi_2 \cap A \neq \emptyset) \leq M_n(A) \cdot \Lambda_2(A)$.
If we show $M_n(A) \longrightarrow 0$, $n \to \infty$, we are done.
Clearly, when the intensity function λ is constant (or bdd), then

$$M_n(A) \leq \lambda \cdot 2^{-nd} \longrightarrow 0 \text{ as } n \to \infty.$$

Lemma: $M_n(A) \to 0$ $\forall$ for any non-neg measurable λ and any $A \in \mathcal{B}(\mathbb{R}^d)$ bounded.

<u>Proof (Lemma):</u> Wlog, assume that A is a finite union of of $Q_{k,0}$.
Then $0 \leq M_{n+1}(A) \leq M_n(A)$ and thus $M_n(A) \to \delta$ for some $\delta \geq 0$. To show $\delta = 0$.
If $\delta > 0$, then $\forall n \exists k_n \in \mathbb{Z}^d$ s.t. $\Lambda(Q_{k_n,n}) \geq \delta$.
Since A is a finite union of $Q_{k,0}$, there is one of them s.t. $Q_{k,0}$ contains infinitely many boxes with Λ-measure $\geq \delta$.

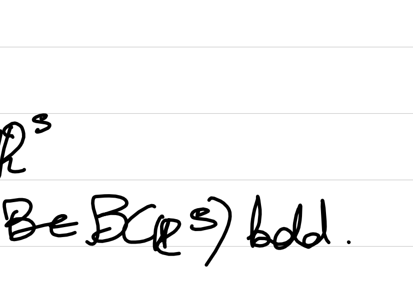
Since Λ is monotone, such a box contains an n-box of Λ-measure $\geq \delta$ $\forall n$.
Call such a box black, i.e., we colour a box $Q_{k,n}$ "black" if $\forall m \geq 0$, $\exists$ a box $Q_{k_m,m} \subset Q_{k,n}$ with $\Lambda(Q_{k_m,m}) \geq \delta$.

Continuing similarly, $Q_{k,0}$ contains some black 1-box, (being a finite union of 1-boxes), and hence obtain a nested seq. of boxes, s.t.
$$Q_0 \supseteq Q_1 \supseteq Q_2 \supseteq \dots$$
s.t. Q_n is an n-box and $\Lambda(Q_n) \geq \delta$.
But $\delta \leq \lim_{n\to\infty} \Lambda(Q_n) \underset{\substack{(MCT)\\(\Lambda(Q_0)<\infty)}}{=} \Lambda(\bigcap_n Q_n) = 0$
$\hookrightarrow$ contains at most one point

That gives a contradiction. $\square$

<u>Rk:</u> The same proof applies to any measure s.t. $\Lambda(\{k\}) = 0 \; \forall k \in \mathbb{R}^d$.

<u>Mapping Theorem:</u> Let $f: \mathbb{R}^d \to \mathbb{R}^s$ be measurable. For Π a Poisson process on $\mathbb{R}^d$, what can we say about the set $f(\Pi) = \{f(x) \in \mathbb{R}^s : x \in \Pi\} \in \mathbb{R}^s$?
When is it a P.P. on $\mathbb{R}^s$?



<u>Thm (Mapping):</u> Let Π be a non-homogeneous P.P. with intensity function λ. Assume that $f: \mathbb{R}^d \to \mathbb{R}^s$ satisfies

(a) $\Lambda(f^{-1}(\{y\})) = 0 \quad \forall y \in \mathbb{R}^s$
(b) $\mu(B) = \Lambda(f^{-1}B) < \infty \quad \forall B \in \mathcal{B}(\mathbb{R}^s)$ bdd.

Then $f(\Pi)$ is a P.P. on $\mathbb{R}^s$ with mean measure μ.

<u>Proof:</u> Assume that the points in $f(\Pi)$ are distinct a.s., i.e. f is injective on Π.
Then, $M(B) = \#\{f(\Pi) \cap B\}$
$= \#\{\Pi \cap f^{-1}(B)\}$ ← f is injective on Π.
$\sim \text{Poi}(\underbrace{\Lambda(f^{-1}B)}_{=\mu(B)})$

if $\forall B_1, B_2, \dots$ are disjoint, so are the preimages $f^{-1}B_1, f^{-1}B_2, \dots$, hence $M(B_i)$ are indep.

Thus, $f(\Pi)$ is a P.P. with mean measure μ.

Thus it suffices to show that the points in $f(\Pi) \cap (0,1]^s$ are distinct a.s. wlog.

Let $Q_{k,n} = \prod_{i=1}^{s} (k_i \cdot 2^{-n}, (k_i+1)\cdot 2^{-n}] \subseteq \mathbb{R}^s$. Then use $(Q_{k,n}, k = 0,1,2,\dots)$ to cover $(0,1]^s$.
Then, for a fixed n,
$N_k^n := \#(\Pi \cap f^{-1}(Q_{k,n})) \sim \text{Poi}(\underbrace{\Lambda(f^{-1}Q_{k,n})}_{=\mu(Q_{k,n}) := \mu_k^n})$

Then, $\mathbb{P}(N_k^n \geq 2) = 1 - e^{-\mu_k^n} - \mu_k^n e^{-\mu_k^n}$
$= 1 - e^{-\mu_k^n}(1 + \mu_k^n)$
$\leq 1 - (1 - \mu_k^n)(1 + \mu_k^n) = (\mu_k^n)^2$.

Then, $\mathbb{P}(N_k^n \geq 2 \text{ for some } k) \leq \sum_k (\mu_k^n)^2$

$\leq (\max_k \mu_k^n) \cdot \sum_k \mu_k^n$
(k runs over all pts in $\mathbb{Z}^d$ s.t. $Q_{k,n}$ covers $(0,1]^s$.)

Thus, $\mathbb{P}(N_k^n \geq 2 \text{ for some } k) \leq M_n((0,1]^s) \cdot \underbrace{\mu((0,1]^s)}_{<\infty}$

$\longrightarrow 0$ as $n \to \infty$ by prev. lemma and $\mu(\{k\}) = 0 \; \forall k \in \mathbb{R}^s$.

i.e., $\mathbb{P}(\bigcap_n \{N_k^n \geq 2 \text{ for some } k\}) = \lim_{n\to\infty} \mathbb{P}(N_k^n \geq 2 \text{ for some } k)$
$= 0$.

<u>I.e., $f(\Pi) \cap (0,1]^s$ are distinct a.s.</u> $\square$

<u>Example:</u> Let Π be a Poisson process on $\mathbb{R}^2$ with constant rate intensity function λ and let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be the polar coordinate function
$$f(x,y) = f(r,\theta), \; r \geq 0, \; \theta \in [0, 2\pi)$$
$$r = \sqrt{x^2+y^2}, \; \theta = \tan^{-1} y/x, \; (0,0) \mapsto (0,0).$$
$$x = r\cos\theta, \; y = r\sin\theta.$$

f is 1-1, so $f^{-1}\{(r,\theta)\} = \{(x,y)\}$ for some $(x,y) \in \mathbb{R}^2$, so clearly (a) holds. Clearly, also (b) holds, so $f(\Pi)$ is a P.P. on $\mathbb{R}^2$ with mean measure $\mu(B) = \int_{(x,y):(r,\theta)\in B} \lambda \, dx\,dy$
$= \int_{(r,\theta)\in B\cap S} \lambda r \, dr\,d\theta$, where $S = f(\mathbb{R}^2) = [0,\infty) \times [0, 2\pi)$

Thus, $f(\Pi)$ is a P.P. on the strip S with intensity function λr.

<u>Conditioning Property:</u> Recall for a P.P. on $\mathbb{R}^+$ with rate λ, given $N_t = n$, the points $0 \leq J_1 \leq \dots \leq J_n$ are the order stats of n iid $U[0,t]$ r.v's, i.e. the set $\{J_1, \dots, J_n\}$ is distributed as n iid $U[0,t]$ r.v's.

<u>Thm (Conditioning Property):</u>

APPLIED PROBABILITY
LECTURE 23

Conditioning Property:

Thm: Let Π be a P.P. on $\mathbb{R}^d$ with intensity f^r λ and $A \in \mathcal{B}(\mathbb{R}^d)$ be $0 < \underset{\int_A \lambda(x)dx}{\underset{\shortparallel}{\Lambda(A)}} < \infty$.

Then conditional on $\#\{\Pi \cap A\} = n$, the n pts in $\Pi \cap A$ have the same distribution as n pts chosen independently according to the prob. distribution ν, where

$$\nu(B) = \frac{\Lambda(B)}{\Lambda(A)}, \quad B \subseteq A, \ B \in \mathcal{B}(\mathbb{R}^d)$$

i.e. with density fn $\dfrac{\lambda(x)}{\Lambda(A)}$, $x \in A$.

In particular, when Π has constant intensity λ, the endpoints are iid uniform in A.

Rk: One can simulate a P.P. using this property.

Proof: Write $N(B) = \#\{B \cap \Pi\}$. Let $A_1, \dots, A_k$ be a partition of A. Then, with $\sum_{i=1}^k n_i = n$

$\mathbb{P}(N(A_1) = n_1, \dots, N(A_k) = n_k \mid N(A) = n)$

$$= \frac{\prod_{i=1}^k \mathbb{P}(\Pi(A_i) = n_i)}{\mathbb{P}(N(A) = n)}$$

$$= \prod_{i=1}^k \frac{e^{-\Lambda(A_i)}(\Lambda(A_i))^{n_i}}{(n_i)!} \cdot \frac{n!}{e^{-\Lambda(A)}(\Lambda(A))^n}$$

$$= \prod_{i=1}^k \frac{(\Lambda(A_i))^{n_i}}{(n_i)!} \frac{n!}{\Lambda(A)^n}$$

$$= \frac{n!}{n_1! \cdots n_k!} \prod_{i=1}^k \nu(A_i)$$

This multinomial distribution is the same as for n indep. points chosen from ν. As this holds for any partition $A_1, \dots, A_k$ of A with $k \geq 1$, this characterises the conditional dist. of $\Pi \cap A \mid \Pi(A) = n$. □

Colouring Principle: Recall the thinning/colouring thm for P.P.(λ) on $\mathbb{R}^+$, which can be viewed as a complementary result to the superposition theorem.

Thm (Colouring): Let Π be a P.P.(λ) on $\mathbb{R}^d$ with intensity function $\lambda : \mathbb{R}^d \to \mathbb{R}$. Colour the points $x \in \Pi$ independently as follows
- A point $x \in \Pi$ is coloured red with prob $\gamma(x)$
- A point $x \in \Pi$ is coloured blue with p. $\underset{:= \sigma(x)}{(1-\gamma(x))}$

indep. of Π, where $\gamma : \mathbb{R}^d \to [0, 1]$.

Let $\Gamma^{\subseteq \Pi}$ and $\Sigma^{\subseteq \Pi}$ be the set of red and blue points respectively. Then Γ and Σ are indep. P.P. on $\mathbb{R}^d$ with intensity f^n's $\gamma(x)\lambda(x)$ and $\sigma(x)\lambda(x)$ resp.

Proof: Let $A \in \mathcal{B}(\mathbb{R}^d)$ with $\Lambda(A) < \infty$. Condition of $\#\{\Pi \cap A\} = n$. Then $\Pi \cap A$ consists of n points chosen indep. from dist^n ν (where $\nu(B) = \frac{\Lambda(B)}{\Lambda(A)}$ $\forall B \subseteq A$, $B \in \mathcal{B}(\mathbb{R}^d)$)

By indep. of pts, their colours are indep. of one another. The prob. that a given pt is coloured red is $\bar{\gamma} = \frac{1}{\Lambda(A)} \int \gamma(x) \lambda(x) dx$.

Let N_r and N_b be the # of red & blue pts in A, then $N_r \mid N(A) = n \sim \text{Bin}(n, \bar{\gamma})$, i.e.,

$\mathbb{P}(N_r = n_r, N_b = n_b \mid N(A) = n) = \binom{n}{n_r} \bar{\gamma}^{n_r} (1-\bar{\gamma})^{n_b}$
$(n_r + n_b = n)$

$$\Rightarrow \mathbb{P}(N_r = n_r, N_b = n_b) = \mathbb{P}(N(A) = n) \times \binom{n}{n_r} \bar{\gamma}^{n_r} (1-\bar{\gamma})^{n_b}$$

$$= e^{-\Lambda(A)} \frac{(\Lambda(A))^n}{\cancel{n!}} \frac{\cancel{n!}}{n_r! n_b!} \bar{\gamma}^{n_r} (1-\bar{\gamma})^{n_b}$$

$$= \frac{(\bar{\gamma}\Lambda(A))^{n_r}}{n_r!} e^{-\bar{\gamma}\Lambda(A)} \cdot \frac{((1-\bar{\gamma})\Lambda(A))^{n_b}}{n_b!} \times e^{-(1-\bar{\gamma})\Lambda(A)}$$

Thus, $\underset{= \#\{\Gamma \cap A\}}{N_r}$ and $\underset{= \#\{\Sigma \cap A\}}{N_b}$ are indep. Poi$(\bar{\gamma}\Lambda(A))$ and Poi$((1-\bar{\gamma})\Lambda(A))$ resp., where $\bar{\gamma}\Lambda(A) = \int_A \gamma(x)\lambda(x)dx$.

The indep. of # red/blue pts in disjoint sets follows from the corresponding prop of Π. □

Applications:

Olbers Paradox: (Heinrich Wilhelm Olbers): Suppose stars occur in $\mathbb{R}^3$ as the pts of a P.P. Π on $\mathbb{R}^3$ with constant intensity λ.
For $x \in \Pi$, let B_x be the brightness of the star at x, and B_x are iid with mean β. The intensity of light striking the observer at the origin O from a star of brightness B, distance r away from O, is given by $\frac{cB}{r^2}$ for some constant $c > 0$. Hence, the total intensity of light at O, from all stars within distance a from O is

$$I_a = \sum_{\substack{x \in \Pi, \\ |x| \leq a}} \frac{c B_x}{|x|^2}, \text{ where } |x| \text{ is the Euclidean distance of } x \text{ from } O.$$

Let $N_a = \#\{\Pi \cap B_a(0)\}$ where $B_a(0)$ is the ball of radius a around O.
Then, given $N_a = n$, the n stars are iid uniform on $B_a(0)$. Thus,

$$\mathbb{E}[I_a \mid N_a] = N_a \cdot c \cdot \beta \, \mathbb{E}\left[\frac{1}{|U|^2}\right], \text{ where } U \sim \text{Unif}(B_a(0)).$$

$$= N_a \cdot c \cdot \beta \cdot \frac{1}{|B_a|} \int_{B_a} \frac{1}{|x|^2} dx$$

Since $N_a \sim \text{Poi}(\lambda |B_a(0)|)$, so, $\mathbb{E}N_a = \lambda |B_a(0)|$,

so $$\mathbb{E}[I_a] = \mathbb{E}N_a \cdot c \cdot \beta \cdot \frac{1}{|B_a(0)|} \int_{|B_a|} \frac{1}{|x|^2} dx$$
$$= c \cdot \lambda \cdot \beta \cdot \int_{|B_a|} \frac{1}{|x|^2} dx$$
$$= c \cdot \lambda \cdot \beta \cdot a \cdot 4\pi$$
$$= \lambda c \beta \cdot 4\pi a.$$

In particular, as $a \to \infty$, $\mathbb{E}[I_a] \to \infty$. This is 'Olbers Paradox' suggests the night sky should be infinitely bright!

Example: A museum has n different rooms that the visitors have to view in sequence. Assume visitors arrive according to a P.P. on $\mathbb{R}^+$ with constant intensity λ. The r-th visitor spends time $X_{r,s}$ in the room s, where $X_{r,s}$ $r \geq 1$, $1 \leq s \leq n$ are indep. and given s, the dist. of $X_{r,s}$, $r \geq 1$ does not depend on r. Let $V_s(t)$ be the # visitors in room s at time t, then show that for any fixed $t > 0$, $V_s(t)$, $s = 1, 2, \dots, n$ are indep. Poisson r.v.'s.

<u>Renyi's Theorem:</u> Complete information of a P.P. is captured by the "void probabilities."

<u>Thm:</u> Let Π be a random countable subset of $\mathbb{R}^d$, and let $\lambda : \mathbb{R}^d \to \mathbb{R}$ be a non-neg. measurable function with $\Lambda(A) = \int_A \lambda(x)\,dx < \infty$ $\forall$ bdd $A \in \mathcal{B}(\mathbb{R}^d)$.

If $\mathbb{P}(\Pi \cap A = \emptyset) = e^{-\Lambda(A)}$ $\circledast$ for any A that is a finite union of n-boxes

$$Q_{k,n} = \prod_{i=1}^{d} (k_i \cdot 2^{-n}, (k_i+1)\cdot 2^{-n}], \quad k=(k_1,\dots,k_d)\in\mathbb{Z}^d, \; n \in \mathbb{N}$$

then Π is a Poisson process with intensity function λ.

<u>Proof:</u> Let $A \subseteq \mathbb{R}^d$ be any bdd open. Let $I_{k,n} = \mathbb{1}(Q_{k,n} \cap \Pi \neq \emptyset)$.

Then, $N(A) = \#\{\Pi \cap A\} = \lim_{n\to\infty} \underbrace{\sum_{k, Q_{k,n} \subseteq A} I_{k,n}}_{N_n(A)}$ (open)

Note $N_n(A) \le N_{n+1}(A)$, so the limit is monotone.

Also, $\Lambda(A) = \lim_{n\to\infty} \sum_{k,\, Q_{k,n} \subseteq A} \Lambda(Q_{k,n})$

Also for a fixed n, $I_{k,n}$'s are indep.

Indeed, $\mathbb{P}(I_{k_1,n}=0, I_{k_2,n}=0)$

$$= \mathbb{P}(Q_{k_1,n} \cap \Pi = \emptyset, Q_{k_2,n} \cap \Pi = \emptyset)$$
$$= \mathbb{P}((Q_{k_1,n} \cup Q_{k_2,n}) \cap \Pi = \emptyset)$$
$$= e^{-\Lambda(Q_{k_1,n} \cup Q_{k_2,n})} \quad \text{by } \circledast$$
$$= e^{-\Lambda(Q_{k_1,n})} e^{-\Lambda(Q_{k_2,n})}$$
$$= \mathbb{P}(I_{k_1,n}=0)\,\mathbb{P}(I_{k_2,n}=0).$$

For a fixed n, as $I_{k,n}$ are indep, for $s \in (0,1)$ small enough,

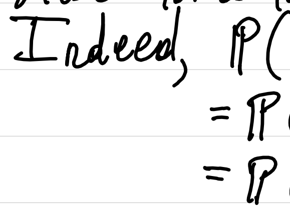
$$E[s^{N_n(A)}] = \prod_{k: Q_{k,n}\subseteq A} E[s^{I_{k,n}}]$$
$$= \prod_{k: Q_{k,n}\subseteq A} \left(e^{-\Lambda(Q_{k,n})} + s(1-e^{-\Lambda(Q_{k,n})})\right)$$
$$= \prod_{k: Q_{k,n}\subseteq A} \left(s + (1-s)e^{-\Lambda(Q_{k,n})}\right)$$

By MCT, $E[s^{N(A)}] = \lim_{n\to\infty} E[s^{N_n(A)}]$

$$= \lim_{n\to\infty} \prod_{k: Q_{k,n}\subseteq A} \left(s + (1-s)e^{-\underbrace{\Lambda(Q_{k,n})}_{:=\alpha}}\right)$$

To show, $s+(1-s)e^{-\alpha} \approx e^{-(1-s)\alpha}$

$$e^{-(1-s)\alpha} \le s + (1-s)e^{-\alpha} \le e^{-(1-s)\alpha + O(\alpha^2)}$$



$f(x) = e^{-\alpha x}$ convex, $\alpha \ge 0$.

For second inequality, enough to show $\log(s+(1-s)e^{-\alpha}) \le -(1-s)\alpha + O(\alpha^2)$

$$\log(s+(1-s)e^{-\alpha}) = -\alpha + \log((1-s)+se^{\alpha})$$
$$= -\alpha + \log(1+s(e^{\alpha}-1))$$

$(e^x \ge 1+x)$

$$\le -\alpha + s(e^{\alpha}-1)$$
$$= -(1-s)\alpha + O(\alpha^2).$$

$$\Rightarrow \lim_{n\to\infty} e^{-(1-s)\sum_{k,\,Q_{k,n}\subseteq A}\Lambda(Q_{k,n})} \le E[s^{N(A)}] \le \lim_{n\to\infty} e^{-(1-s)\sum_{k:Q_{k,n}\subseteq A}(\Lambda(Q_{k,n}) + O(\Lambda^2))}$$

Now, $\sum_{k: Q_{k,n}\subseteq A} \Lambda^2(Q_{k,n}) \le \underbrace{\max_k \Lambda(Q_{k,n})}_{M_n(A)} \underbrace{\left(\sum_k \Lambda(Q_{k,n})\right)}_{\le \Lambda(A)}$

$$\le \Lambda(A)\cdot M_n(A) \to 0 \text{ as } n\to\infty.$$

Also recall, $\Lambda(A) = \lim_{n\to\infty} \sum_{k: Q_{k,n}\subseteq A} \Lambda(Q_{k,n})$

Hence, $E[s^{N(A)}] = e^{-(1-s)\Lambda(A)}$, i.e., $N(A) \sim \mathrm{Poi}(\Lambda(A))$.

Also, $N(A_1),\dots,N(A_k)$ are ind. for disjoint open sets $A_1,\dots,A_k$ because the $N_n(A_i)$ are indep. for $i=1,\dots,k$ and $N_n(A_i) \to N(A_i)$ as $n\to\infty$.

By a standard approximation argument, the statement can be extended to any A bdd to general $\in \mathcal{B}(\mathbb{R}^d)$.